Problem
Find all solutions to

$$b - a = c - b = ac - d = \sqrt{d - a - b - c}$$

where a, b, c, d are primes. Show that the systems

$$b - a = c - b = ac - d$$
$$b - a = c - b = \sqrt{d - a - b - c}$$

each admit more solutions in primes than the first.

Solution
It is easy to verify that $(3, 5, 7, 19)$ is a solution to the first system, while $(3, 7, 11, 29)$ and $(3, 7, 11, 37)$ are solutions to the second and third respectively but not the third and second respectively.

Let

$$s = b - a = c - b = ac - d = \sqrt{d - a - b - c}$$

for convenience.

Since d is prime, $s = ac - d \neq 0$. Since $s = \sqrt{d - a - b - c} \geq 0$, it must be true that $s > 0$ Then $a < b < c$.

Note that $2b = a + c$. Then $a \neq 2$, for if it were then c would be an even prime > 2, which is absurd. So a is odd, and since b, c are primes $> a$ they are also odd. Then $s = ac - d$ is even, and so d is odd.

Now s is congruent to one of $0, 2, 4 \pmod 6$ since it is even and a is congruent to one of $1, 3, 5 \pmod 6$ because it is odd. The only prime equivalent to 3 is 3, and since 3 is the least odd prime and $a < b < c$, b and c are never equivalent to 3. Then the two broad cases are

$$a = 3 \text{ and } s \equiv 2, 4 \pmod 6$$
$$a \equiv 1, 5 \pmod 6 \text{ and } s \equiv 0 \pmod 6$$

Suppose towards contradiction the latter case is true. Then $\sqrt{d - a - b - c} = s \equiv 0 \pmod 6$ and $a + b + c \equiv 3 \pmod 6$. Substituting and squaring, we have $d - 3 \equiv 0 \pmod 6$ and so $d = 3$. But then $d - a - b - c < 0$, so the square root is not defined; contradiction.

So $a = 3$ and $s \equiv 2, 4 \pmod 6$. Then $d - a - b - c = s^2 \equiv 4 \pmod 6$ and since $a + b + c \equiv 3 \pmod 6$, $d \equiv 1 \pmod 6$. Now $s = ac - d \equiv 3c - 1 \equiv 2 \pmod 6$.

Write $s = 6t + 2$ and note that $a = 3, b = 6t + 5, c = 12t + 7$. Furthermore $d = ac - s = 3(12 + 7) - (6t + 2) = 30t + 19$. Also, $6t + 2 = \sqrt{d - a - b - c} = \sqrt{12t + 4}$, so $2(3t + 1) = 2\sqrt{3t + 1}$. Then $9t^2 + 6t + 1 = 3t + 1 \implies 3t(3t + 1) = 0$, and the only integer solution for t is 0. This yields the single solution $(3, 5, 7, 19)$.