Problem

Let a be the positive 2020-th root of 0.99 and let $f : \mathbb{R} \to \mathbb{R}$ satisfy the functional equation

$$f(f(x)) = 2af(x) - a^2x$$

Prove that $f(0) = 0$ and find a non-zero solution for f.

Solution

First we show by strong induction that

$$f^n(x) = na^{n-1}f(x) - (n-1)a^nx$$

This holds for $n = 1$ and $n = 2$. Suppose it holds for all $k \leq n$ for some $n \geq 2$. Then

$$\begin{aligned}
f^{n+1}(x) &= f^2(f^{n-1}(x)) \\
&= 2af(f^{n-1}(x)) - a^2f^{n-1}(x) \\
&= 2af^n(x) - a^2f^{n-1}(x) \\
&= 2a(na^{n-1}f(x) - (n-1)a^nx) - a^2((n-1)a^{n-2}f(x) - (n-2)a^{n-1}x) \\
&= 2na^nf(x) - 2(n-1)a^{n+1}x - (n-1)a^nf(x) + (n-2)a^{n+1}x \\
&= (n+1)a^nf(x) - na^{n+1}x
\end{aligned}$$

as desired. In particular, $f^n(0) = (n+1)f(0)$.

Now, $\lim_{n\to\infty}(n+1)a^n = 0$ and $\lim_{n\to\infty} na^{n+1} = 0$ because $a < 1$. So for all x, $\lim_{n\to\infty} f^n(x) = 0$.

Let $c = 99 * 2020 - 1$ and calculate that

$$\begin{aligned}
f^c(0) &= (c+1) * a^cf(0) \\
&= 0.99 * (c+2021) * a^cf(0) \\
&= (c+2021)a^{c+2021}f(0) = f^{c+2020}(0)
\end{aligned}$$

Indeed, it is easy to prove by induction that

$$f^{c+2020k}(0) = f^{2020k}(f^c(0)) = f^c(0) = (c+1) * a^cf(0)$$

for all $k \in \mathbb{N}$. Therefore $f^c(0)$ is a cluster point of the sequence $\{f^n(0)\}$. But since $\lim_{n\to\infty} f^n(0) = 0$, $(c+1)a^cf(0) = f^c(0) = 0$. Finally, since $(c+1)a^c \neq 0$, $f(0) = 0$.