Problem

Let A, B be $n \times n$ matrices over a field $\Bbbk$ of characteristic $\neq 3$ such that $AB = A^2 + B^2$ and $AB - BA$ is invertible. Prove that n is divisible by 3.

Solution

Let $\Bbbk$ be a field of characteristic $\neq 3$ and let ω be a primitive 3rd root of unity in $\overline{\Bbbk}$. Since $\omega^2 + \omega + 1 = 0$,

$$\begin{aligned}
(A + \omega B)(A + \omega^2 B) &= A^2 + B^2 + \omega^2 AB + \omega BA \\
&= A^2 + B^2 - (1 + \omega)AB + \omega BA \\
&= (A^2 + B^2 - AB) - \omega(AB - BA) \\
&= -\omega(AB - BA) \\
(A + \omega^2 B)(A + \omega B) &= A^2 + B^2 + \omega AB + \omega^2 BA \\
&= A^2 + B^2 - (1 + \omega^2)AB + \omega^2 BA \\
&= (A^2 + B^2 - AB) - \omega^2(AB - BA) \\
&= -\omega^2(AB - BA)
\end{aligned}$$

Let $d = \det(AB - BA)$ and note that $\det(-\omega(AB - BA)) = (-1)^n \omega^n d$ and $\det(-\omega^2(AB - BA)) = (-1)^n \omega^{2n} d$. Since the determinant is multiplicative and multiplication in $\Bbbk$ is commutative,

$$\begin{aligned}
\det(-\omega(AB - BA)) &= \det((A + \omega B)(A + \omega^2 B)) \\
&= \det(A + \omega B) \det(A + \omega^2 B) \\
&= \det(A + \omega^2 B) \det(A + \omega B) \\
&= \det((A + \omega^2 B)(A + \omega B)) = \det(-\omega^2(AB - BA))
\end{aligned}$$

Therefore $(-1)^n \omega^n d = (-1)^n \omega^{2n} d$ and since $AB - BA$ is invertible, $d \neq 0$ and $\omega^n = \omega^{2n}$. Then $\omega^n = 1$ since ω^n is a non-zero solution to $x(x - 1) = x^2 - x = 0$. Since ω is a *primitive* 3rd root of unity, $3 \mid n$.