Nova Scotia Math League

2010–2011

Game One

SOLUTIONS

Team Question Solutions

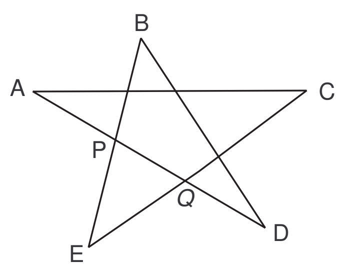
1. Since $108 = 2^2 \cdot 3^3$, its divisors are all numbers of the form $2^i 3^j$, where $0 \le i \le 2$ and $0 \le j \le 3$. The sum of all such numbers is $(2^0 + 2^1 + 2^2)(3^0 + 3^1 + 3^2 + 3^3) = 7 \cdot 40 = 280$.

2. Since the terms are in geometric progression we must have

$$\frac{2x+4}{x} = \frac{3x+6}{2x+4} = \frac{3(x+2)}{2(x+2)} = \frac{3}{2}.$$

Thus the common ratio is $\frac{3}{2}$, and solving $\frac{2x+4}{x} = \frac{3}{2}$ yields $x = -8$. The next term is therefore $(-8)(\frac{3}{2})^3 = -27$.

3. The given angles are irrelevant. Let points P and Q be as indicated in the diagram below, and let a, b, c, d, e be the measures of the acute angles at A, B, C, D, E, respectively. Then $\angle EPQ$ is exterior to $\triangle PBD$, so that $\angle EPQ = b + d$. Similarly, $\angle EQP = a + c$. Hence $a + b + c + d + e = \angle EPQ + \angle EQP + e$, and this sum is 180° because it is the sum of the angles of $\triangle EPQ$.



4. Since a and b are real, the fact that $1 - 2i$ is a root forces the other root to be its conjugate, namely $1 + 2i$. Moreover, we know $-\frac{a}{3}$ is the sum of the roots, and $\frac{b}{3}$ is their product. That is,

$$-\frac{a}{3} = (1-2i) + (1+2i) = 2$$
$$\frac{b}{3} = (1-2i)(1+2i) = 5.$$

This gives $a = -6$ and $b = 15$.

Alternative Solution: Substitute $x = 1 - 2i$ into $3x^2 + ax + b = 0$ to obtain

$$(a + b - 9) - (12 + 2a)i = 0.$$

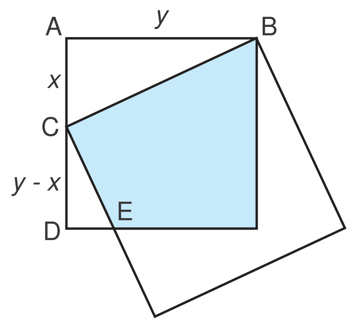
Hence $a + b - 9 = 0$ and $12 + 2a = 0$, from which we get $a = -6$ and $b = 15$.

5. Let n be the total number of votes cast. Then the problem suggests that

$$\frac{25 + \frac{1}{5}(n - 50)}{n} = \frac{1}{4},$$

which is readily solved to obtain $n = 300$.

6. Consider the general situation below, with x, y and points A through E as indicated. Since $\angle BCE = 90°$, we know $\angle ACB$ and $\angle DCE$ are complementary, and from this it follows that right triangles $\triangle ABC$ and $\triangle DCE$ are similar, with $\triangle DCE$ smaller by a factor of $\frac{y-x}{y}$. Thus the area of $\triangle DCE$ is $(\frac{y-x}{y})^2$ times that of $\triangle ABC$.



The area of the shaded region is then

$$\text{Area}(\Box ABCD) - \text{Area}(\triangle ABC) - \text{Area}(\triangle DCE) = y^2 - \frac{xy}{2} - \left(\frac{y-x}{y}\right)^2 \cdot \frac{xy}{2}.$$

In our particular instance, the small and large squares have areas 25 and 29. Thus $y = 5$ and $|BC| = \sqrt{29}$. The Pythagorean theorem on $\triangle ABC$ gives $x^2 + y^2 = |BC|^2$, or $x^2 + 5^2 = 29$, which yields $x = 2$. Finally then, we set $x = 2$ and $y = 5$ in the expression above to find that the shaded area is $\frac{91}{5}$.

7. Let $d(n)$ be the number of ways of obtaining a total of *at most* n when rolling two dice. Then the desired probability is $\frac{1}{216}(d(1) + d(2) + \cdots + d(6))$, since there are $6^3 = 216$ possible rolls of the three dice, and for each value n of the red die there are $d(n)$ valid configurations of the blue dice.

   Recall that for $i = 1, 2, \ldots, 6$ there are $i - 1$ ways ways of obtaining a total of i when rolling a pair of 6-sided dice. Thus for $n = 1, 2, 3, \ldots, 6$, we have $d(n) = \sum_{i=1}^{n}(i-1)$. Thus $d(1) = 0$, $d(2) = 1$, $d(3) = 1 + 2 = 3$, $d(4) = 1 + 2 + 3 = 6$, $d(5) = 1 + \cdots + 4 = 10$, and $d(6) = 15$. The desired probability is then $\frac{1}{216}(1 + 3 + 6 + 10 + 15) = \frac{35}{216}$.

   Note: Consider the same problem in the more general setting of throwing three m-sided dice. (That is, the problem above is the special case $m = 6$.) Since the answer with 6-sided dice is $(6^2 - 1)/6^3$, it is tempting to guess that the probability in the case of m-sided dice is $(m^2 - 1)/m^3$. But it turns out this isn't true!

   As before, let $d(n)$ be the number of ways of obtaining a total of at most n when rolling two m-sided dice. Then we again get $d(n) = \sum_{i=1}^{n}(i-1)$, and this sum can be computed to give the closed-form expression $d(n) = \frac{1}{2}n(n-1)$, valid for $1 \le n \le m$. From here we can deduce that $\sum_{n=1}^{m} d(n) = \frac{1}{6}m(m^2 - 1)$. This counts the number of rolls of three m-sided dice — two blue and one red — such that the red die comes up at least as great as the sum of the blues. Since there are m^3 rolls in total, the probability of such a roll is $\frac{1}{m^3} \cdot \frac{1}{6m^2} = \frac{m^2-1}{6m^2}$. This is the correct general formula! Note that the '6' appears in the denominator regardless of m. It is just a fluke that the denominator is m^3 when $m = 6$.

8. Upon expanding $(x - 2y + 3z)^4$, we obtain the sum of the coefficients simply by setting $x = y = z = 1$. Of course, this substitution can equally well be done before expanding. The coefficients therefore sum to $(1 - 2 + 3)^4 = 2^4 = 16$.

   Alternative Solution: Compute $(x - 2y + 3z)^4$ by first squaring $x - 2y + 3z$, and then squaring the result. It's a fair bit of work!

9. Let the square have sides of length x, and let the rectangle have width y and length $3y$ (all lengths in metres). Since the total perimeter of the two shapes must be 4, we have $4x + 2(y + 3y) = 4$, or simply $x + 2y = 1$. Note that $0 \leq x \leq 1$ and $0 \leq y \leq \frac{1}{2}$.

   Now the sum of the areas of the shapes is given by

   $$A = x^2 + 3y^2 = (1 - 2y)^2 + 3y^2 = 1 - 4y + 7y^2.$$

   We wish to minimize A subject to the condition that $0 \leq y \leq \frac{1}{2}$. Since A is quadratic in y, its minimum will occur at the average of its roots, namely when $y = \frac{4}{2 \cdot 7} = \frac{2}{7}$. At this value of y we have $A = \frac{3}{7}$.

10. Every valid arrangement can be obtained as follows: First enter 1 in any of the cells. This can be done in nine ways. There are now precisely four cells in which the 9 can be placed. After doing so, there are seven cells remaining. Simply enter $8, 7, 6, 5, 4, 3, 2$ amongst these cells without restriction. This can be done in $7 \cdot 6 \cdot 5 \cdot \cdots \cdot 2 \cdot 1$ ways. There are therefore $9 \cdot 4 \cdot 7! = 36 \cdot 7!$ possible arrangements.

Pairs Relay Solutions

A. Suppose the polygon has n sides. Then the sum of all internal angles is $144n$ degrees. But the sum of the interior angles of a n-gon is always $180(n-2)$ degrees, so $144n = 180(n-2)$. Solve to get $n = 10$. So $\mathtt{A} = 10$.

B. If net effect of the stated year-over-year changes is a factor of

$$\frac{125}{100} \cdot \frac{140}{100} \cdot \frac{80}{100} \cdot \frac{100 - \mathtt{A}}{100} = \frac{7(100 - \mathtt{A})}{500}.$$

Since $\mathtt{A} = 10$, this evaluates to $\frac{63}{50} = \frac{126}{100}$. So the net percent increase is 26%. Thus $\mathtt{B} = 26$.

C. Observe that $1! = 1, 2! = 2, 3! = 6, 4! = 24, 5! = 120$. After this point, all factorials must end with a 0. So, unless $\mathtt{B} < 5$, the desired digit will be the same as the units digit of $1 + 2 + 6 + 24 = 33$, namely 3. Indeed, $\mathtt{B} = 26$, so we get $\mathtt{C} = 3$.

D. Since $PQRS$ is a parallelogram, the midpoint of diagonal PR must coincide with the midpoint of diagonal QS. Thus we have $(\frac{1}{2}(1+5), \frac{1}{2}(\mathtt{C}+1)) = (\frac{1}{2}(-1+x), \frac{1}{2}(-2+y))$. Thus $x = 7$ and $y = \mathtt{C} + 3$.

Since $\mathtt{C} = 3$, we have $\mathtt{D} = x + y = 7 + \mathtt{C} + 3 = 13$.

Individual Relay Solutions

A. We have $54\texttt{A} = 2 \cdot 3^3 \cdot \texttt{A}$. The smallest $\texttt{A}$ for which this is a perfect square is therefore $\texttt{A} = 2 \cdot 3 = 6$.

(*Note:* An integer is a perfect square if and only if contains only even powers of primes.)

B. Let S be the sum of the original set of numbers and let $\texttt{B}$ be the missing number. Then we have $S/\texttt{A} = 17$ and $(S - \texttt{B})/(\texttt{A} - 1) = 18$. The first equation gives $S = 17\texttt{A}$, and the second can then be solved to yield $\texttt{B} = 18 - \texttt{A}$.

With $\texttt{A} = 6$ we get $\texttt{B} = 12$.

C. A moment's exploration shows that 1 lies opposite 26, 2 lies opposite 27, 3 lies opposite 28, etc. In general, $\texttt{B}$ and $25 + \texttt{B}$ are diametrically opposite for all $1 \leq \texttt{B} \leq 25$.

Since $\texttt{B} = 12$, we get $\texttt{C} = 25 + 12 = 37$.

D. For a given positive integer y, the equation $2x + 3y = \texttt{C}$ is satisfied for some positive x provided that $\texttt{C} - 3y$ is an even positive integer. But $\texttt{C} - 3y$ is positive provided $y < \texttt{C}/3$, and $\texttt{C} - 3y$ is even if and only if $\texttt{C}$ and y are of the same parity. Therefore $\texttt{D}$ is simply the number of integers y with $1 \leq y < \frac{1}{3}\texttt{C}$ such that y and $\texttt{C}$ are of the same parity.

Since $\texttt{C} = 37$, we require odd integers y between 1 and 12 (inclusive). There are only 6 of these ($y = 1, 3, 5, 7, 9, 11$), so $\texttt{D} = 6$.