NSML

Nova Scotia Math League

2011–2012

Game Three

SOLUTIONS

Team Questions

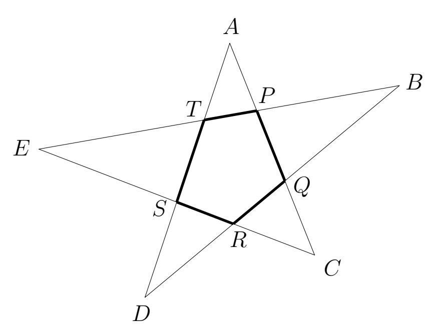
1. Let x be the total number of people in line (including the observant man). Then $x = 1 + \frac{5}{6}x + \frac{1}{7}x$. Solve for x to get $x = 42$.

2. Let ℓ be the length of the ladder. Then the base of the ladder is originally $\ell - 8$ feet from the foot of the wall, so Pythagorean theorem gives $(\ell - 8)^2 + 12^2 = \ell^2$. Solve for ℓ to get $\ell = 13$.

3. There are 9 one-digit palindromes (namely $1, 2, \ldots, 9$), and also 9 two-digit palindromes (namely $11, 22, \ldots, 99$). There are $9 \cdot 10 = 90$ three digit palindromes, since the first digit may be 1 through 9 and the second can be 0 through 9, while the third digit is the same as the first. Similarly, there are 90 four-digit palindromes, since the first two digits can be chosen in $9 \cdot 10 = 90$ ways, and the final two digits are then determined. Thus there are $9 + 9 + 90 + 90 = 198$ palindromes less than 100000 altogether.

4. Let the small circles each have radius 1, so that their total area is 7π. Then the large circle has radius 3 and area 9π. The desired ratio is therefore $7\pi/9\pi = 7/9$.

5. Let points A through E and P through T be as indicated in the figure below. Then summing the interior angles of the five triangles $\triangle TBD$, $\triangle PCE$, $\triangle QDA$, $\triangle REB$ and $\triangle SAC$ yields twice the desired sum, plus the sum of the interior angles of pentagon $PQRST$. Letting x be the desired sum, we therefore have $5 \cdot 180 = 2x + 3 \cdot 180$. Therefore $x = 180°$.



6. Without loss of generality, assume Alice walks at 1 km/h and suppose Alice and Bob have each walked for x hours when they meet at 11am. In those first x hours, Bob covers the 4 km of trail that Alice will walk between 11am and 3pm. His speed is therefore $\frac{4}{x}$ km/h. But in the 9 hours between 11am to 8pm, Bob will cover the x km of trail that Alice walked between sunrise and 11am, making his speed $\frac{x}{9}$. Setting $\frac{4}{x} = \frac{x}{9}$ yields $x = 6$, so we conclude that sunrise was 6 hours before 11am.

7. Let $S_{\text{odd}} = a_1 + a_3 + a_5 + \cdots + a_{2011}$ and $S_{\text{even}} = a_2 + a_4 + a_6 + \cdots + a_{2012}$. Since $a_i =$

$a_{i+1} - 2$, we have

$$\begin{aligned} S_{\text{odd}} &= a_1 + a_3 + a_5 + \cdots + a_{2011} \\ &= (a_2 - 2) + (a_4 - 2) + (a_6 - 2) + \cdots + (a_{2011} - 2) \\ &= a_2 + a_4 + a_6 + \cdots a_{2012} - 2 \cdot 1006 \\ &= S_{\text{even}} - 2 \cdot 1006. \end{aligned}$$

But we also know $S_{\text{even}} + S_{\text{odd}} = 10000$, and it follows that $S_{\text{even}} = 6006$.

8. A product of integers is odd if and only if each of its factors is odd. Since there are 5 odd numbers in $\{1, 2, 3, \ldots, 10\}$, the probability of an odd product is

$$\frac{5 \cdot 4 \cdot 3}{10 \cdot 9 \cdot 8} = \frac{1}{12}.$$

The probability of an even product is therefore $1 - \frac{1}{12} = \frac{11}{12}$.

9. Let α be the number chosen by your friend. Here is an mechanism to guess the correct value of α in 14 (or fewer) guesses:

- First guess is 14.
  - If $\alpha = 14$, you win in one guess. Done!
  - If $\alpha < 14$, proceed to guess $1, 2, 3, \ldots, 13$, in order. You are guaranteed to find α by doing so, but this may require 13 additional guesses. Therefore you will find α in a total of at most $1 + 13 = 14$ guesses. Done!
  - If $\alpha > 14$, move to the next step.
- You've made 1 guess so far. Next guess is $27 = 14 + 13$.
  - If $\alpha = 27$, you win in a total of two guesses. Done!
  - If $\alpha < 27$, proceed to guess $15, 16, 17, \ldots, 26$, in order. You are guaranteed to find α in a total of at most $2 + 12 = 14$ guesses. Done!
  - If $\alpha > 27$, move to the next step.
- You've made 2 guesses so far. Next guess is $39 = 14 + 13 + 12$.
  - If $\alpha = 39$, you win in a total of three guesses. Done!
  - If $\alpha < 39$, proceed to guess $28, 29, 30, \ldots, 38$, in order. You are guaranteed to find α in a total of at most $3 + 11 = 14$ guesses. Done!
  - If $\alpha > 39$, move to the next step.

Continue on in this pattern, "forward guessing" and then "backfilling" if your guess was too high. (Your next forward guess is $50 = 14 + 13 + 12 + 11$, etc.) Since $14 + 13 + 12 +$

$\cdots + 3 + 2 + 1 > 100$, you are guaranteed to make a forward guess that is at least as big as α in 14 or fewer steps. And, by design, the backfilling stage will never push you over a total of 14 guesses.

Now that we have see that it is possible to guarantee a win in 14 or fewer guesses, we must prove that it is *not* possible to guarantee a win in fewer guesses. (That is, 14 is the best we can do.) This is an instrumental part of a complete solution, but the analysis is a very good logical exercise and it is left to you to work through the details on your own.

Note: Of course, there is nothing special about 100. If you were instead trying to guess a number between 1 and n (according to the same rules), then a similar analysis shows that you would need at most k guesses, where k is the smallest positive integer such that

$$1 + 2 + 3 + \cdots + k \geq n.$$

Using $1 + 2 + \cdots + k = \frac{1}{2}k(k+1)$, this inequality can be rewritten as $k^2 + k - 2n \geq 0$. The quadratic formula then yields

$$k = \left\lceil \frac{-1 + \sqrt{1 + 8n}}{2} \right\rceil.$$

As usual, $\lceil x \rceil$ (the "ceiling" of x) denotes the least integer greater than or equal to x.

Pairs Relay

P-A. There are 6 possible arrangements: TEST, TSET, ETST, STET, TETS, and TSTE.

P-B. The equation $3x + 4y = 36$ has exactly 4 nonegative integer solutions, namely $(12, 0)$, $(8, 3)$, $(4, 6)$, and $(0, 9)$.

P-C. Multiply the equations $xy = 1$, $yz = 9$, and $zx = 4$ to get $(xyz)^2 = 36$. Since x, y, z are positive, obtain $xyz = 6$.

P-D. Let $x = 1/\mathtt{D}$ so that the equation becomes

$$\frac{1}{x(1+x)} = \frac{1}{\mathtt{C}(\mathtt{C}-1)}.$$

Provided $\mathtt{C} - 1$ is positive, it follows that $x = \mathtt{C} - 1$. Thus $\mathtt{D} = 1/x = 1/(\mathtt{C} - 1) = 1/5$.

Individual Relay

I-A. Divide the region into triangles and calculate areas as usual, or calculate the unshaded area (which works out to 26) and subtract from the area of the rectangle (which is $6 \cdot 8 = 48$).

I-B. With $\mathtt{A} = 22$, the average of $\{1, 7, 11, 19, \mathtt{A}\}$ is $x = \frac{1}{5}(1 + 7 + 11 + 19 + 22) = 12$. Since the average of $x - 1$, x and $x + 1$ is x, adding these numbers to a set with average x does not affect that average. So the average remains 12.

I-C. Complete the square to write $x^2 + 6x + \mathtt{B} = (x + 3)^2 + (\mathtt{B} - 9)$. Clearly this is minimized when $x + 3 = 0$, and the minimum value is $\mathtt{B} - 9 = 12 - 9 = 3$.

I-D. The product simplifies to

$$\frac{3}{2} \cdot \frac{4}{3} \cdot \frac{5}{4} \cdots \frac{2\mathtt{C}}{2\mathtt{C} - 1} \cdot \frac{2\mathtt{C} + 1}{2\mathtt{C}} = \frac{2\mathtt{C} + 1}{2}$$

Setting $\mathtt{C} = 3$ yields $7/2$.