Nova Scotia Math League

2012–2013

Game Two

SOLUTIONS

Team Question Solutions

1. Kelly was born in the year $y^2 - y$. Since this must lie in the 20th century, we have $1900 \leq y^2 - y < 2000$. Noting that $y^2 - y = y(y-1)$, we quickly arrive at only one option for y, namely $y = 45$. Since $45^2 = 2025$, Kelly will be 45 in the year 2025, meaning she is currently $45 - 12 = 33$ years old in 2013.

2. Sunny travels $\frac{4}{5}$ as fast as Moonbeam. So when Moonbeam has travelled m metres, Sunny has travelled $\frac{4}{5}m + 10$ metres. Setting these quantities equal yields $m = 50$.

3. The probability that Alan guesses the last digit correctly on his first call is $\frac{1}{10}$. The probability that he guesses incorrectly on his first call and correctly on his second is $\frac{9}{10} \cdot \frac{1}{9} = \frac{1}{10}$. So the probability that he will be successful is $\frac{1}{10} + \frac{1}{10} = \frac{1}{5}$.

4. Note the pattern: The term $a + b\sqrt{2}$ is followed by the term $(a + 2b) + (a + b)\sqrt{2}$. Thus $99 + 70\sqrt{2}$ is followed by $(99 + 2 \cdot 70) + (99 + 70)\sqrt{2} = 239 + 169\sqrt{2}$.

   Note: In fact, this is a geometric sequence with common ratio $1 + \sqrt{2}$. Notice that $(a + b\sqrt{2})(1 + \sqrt{2}) = (a + 2b) + (a + b)\sqrt{2}$.

5. Suppose the plot is ℓ metres long and w metres wide. We are given that

$$\ell w = (\ell + 15)(w - 4) = \ell w + 15w - 4\ell - 60$$

   and

$$\ell w = (\ell - 15)(w + 8) = \ell w - 15w + 8\ell - 120.$$

   Cancelling ℓw from each of these equations and rearranging leads to the system of equations $\{15w - 4\ell = 60, -15w + 8\ell = 120\}$, which is readily solved to give $\ell = 45$ and $w = 16$. Thus the area of the plot is $45 \cdot 16 = 720$ square metres.

6. Lockers #1–9 cost 5 cents each to label, lockers #10–99 cost 10 cents, lockers #100–999 cost 15 cents, and lockers #1000–9999 cost 20 cents. Suppose there are n lockers. It is clear that $1000 < n < 10000$, so the total cost (in cents) to label the lockers is

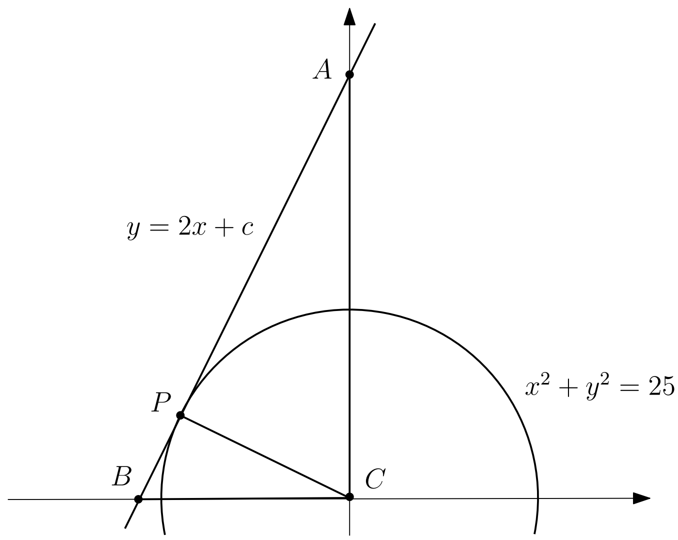
$$9 \cdot 5 + 90 \cdot 10 + 900 \cdot 15 + (n - 999) \cdot 20 = 20n - 5535.$$

   Setting this equal to 34725 yields $n = 2013$

7. Consider the simpler situation in which a square is inscribed in a circle inscribed in a square. Then the side of the outer square equals the diagonal of the inner square, since both quantities are the diameter of the circle. By Pythagorean theorem, this implies the outer square is $\sqrt{2}$ times larger than the inner square, and therefore has twice the area.

We have just found that if you inscribe a square within a circle within a square, then the bigger square has twice the area of the smaller square. So when we repeat this process two more times to arrive at the given diagram, the largest square has area $2 \cdot 2 \cdot 2 = 8$ times larger than the smallest. Thus the desired area is $\frac{9}{8}$.

8. The maximum possible value of c is that which makes the line $y = 2x + c$ tangent to the circle at a point P in the second quadrant, as illustrated below.



Let us suppose this is the case. We wish to find the value of $|AC| = c$.

Since AB is tangent to the circle at P, we know $\angle APC$ is a right angle, and thus triangles $\triangle ABC$ and $\triangle ACP$ are similar. Since line AB has slope 2, we know $|AC| : |BC| = 2 : 1$, and therefore $|AP| : |PC| = 2 : 1$ by similarity. It follows from Pythagorean theorem that $|AC| : |PC| = \sqrt{5} : 1$. But $|PC| = 5$, since this is a radius of the circle. Therefore $|AC| = 5\sqrt{5}$.

Alternate solution: In the above diagram, note that lines AB and CP are perpendicular when AB is tangent to the circle at P. Thus the slope of CP is $-\frac{1}{2}$ (since AB has slope 2), and since CP passes through the origin its equation is $y = -\frac{1}{2}x$. Substituting $x = -2y$ into $x^2 + y^2 = 25$ and solving for y yields $y = \pm\sqrt{5}$. Back substitution then gives $x = \pm\sqrt{20} = \pm 2\sqrt{5}$. Since P is in the second quadrant, we conclude that $P = (-2\sqrt{5}, \sqrt{5})$. But P lies on $y = 2x + c$, so we have $\sqrt{5} = 2(-2\sqrt{5}) + c$, whence $c = 5\sqrt{5}$.

Alternate solution: As above, we seek a value of c which makes $y = 2x + c$ tangent to the circle $x^2 + y^2 = 25$. Thus we require that the equation $x^2 + (2x + c)^2 = 25$ has exactly *one* real solution x. This occurs precisely when its discriminant is 0, which in turn gives a quadratic expression for c that is readily solved to give $c = \pm 5\sqrt{5}$. We choose the larger positive value for c.

9. Let the base, perpendicular, and hypotenuse of the triangle be x, y and z respectively. We are given $x+y=23$ and $x+z=25$, and we know $x^2+y^2=z^2$. The latter equation can be rearranged to give

$$x^2 = z^2 - y^2 = (z-y)(z+y).$$

But subtracting $x+y=23$ from $x+z=25$ yields $z-y=2$, whereas adding and rearranging these equations yields $z+y=48-2x$. Thus we have

$$x^2 = 2(48-2x).$$

Solving this quadratic equation for x gives $x=8$ (the negative root is obviously impermissible). This quickly gives $y=15$ and $z=17$, so the perimeter is $x+y+z = 8+15+17=40$.

10. By polynomial long division, we have

$$\frac{m^3+3}{m+3} = m^2 - 3m + 9 - \frac{24}{m+3}.$$

For integer values of m, the right-hand side will be an integer if and only if $m+3$ divides 24. The largest positive value of m for which this is true is $m=21$.

Alternative (but equivalent!) solution: Let $n = m+3$, so that $m = n-3$. Then

$$\frac{m^3+3}{m+3} = \frac{(n-3)^3+3}{n} = \frac{n^3-9n^2+27n-24}{n} = n^2 - 9n + 27 - \frac{24}{n}.$$

Clearly the largest integer value of n for which this is an integer is $n=24$. Thus $m=21$.

Pairs Relay Solutions

P-A. Since (real) squares are nonnegative, the given sum can be zero only if each summand is zero. Thus $x = 20$, $y = -12$, and $z = 24$. This gives $\mathtt{A} = x + y + z = 32$.

P-B. Each win multiplies the gambler's current stake by $\frac{3}{2}$, while each loss multiplies it by $\frac{1}{2}$. So if he begins with $\mathtt{A}$ dollars, he has $(\frac{3}{2})(\frac{3}{2})(\frac{1}{2})(\frac{1}{2})\mathtt{A} = \frac{9}{16}\mathtt{A}$ dollars after two wins and two losses. Thus he has lost $\mathtt{B} = \frac{7}{16}\mathtt{A}$ dollars at this point. Since $\mathtt{A} = 32$, we get $\mathtt{B} = 14$.

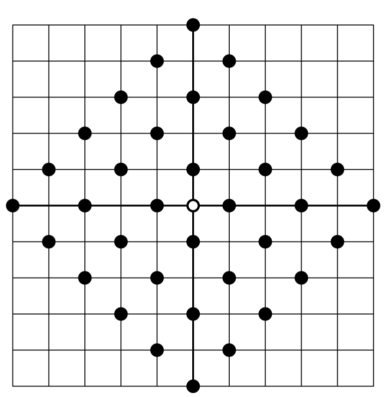
P-C. Let us solve the equation $\mathtt{B} \oplus x = 9$ for x in terms of $\mathtt{B}$:

$$\begin{aligned} \mathtt{B} \oplus x = 9 \quad &\iff \quad \frac{\mathtt{B} + x}{1 + \mathtt{B}x} = 9 \\ &\iff \quad \mathtt{B} + x = 9 + 9\mathtt{B}x \\ &\iff \quad x(1 - 9\mathtt{B}) = 9 - \mathtt{B} \\ &\iff \quad x = \frac{9 - \mathtt{B}}{1 - 9\mathtt{B}}. \end{aligned}$$

Then $\mathtt{C} = \frac{1}{x} = \frac{1-9\mathtt{B}}{9-\mathtt{B}}$. With $\mathtt{B} = 14$, this yields $\mathtt{C} = \frac{-125}{-5} = 25$.

Note: Notice that the solution of the equation $\mathtt{B} \oplus x = 9$ is $x = 9 \oplus (-\mathtt{B})$, which is interesting because it appears that we can solve the equation simply by "balancing", exactly as we would if $\oplus$ was instead the usual $+$ operator. This isn't a coincidence. It turns out that $\oplus$ works an awful lot like regular addition. For instance, it's easy to see that $x \oplus (-x) = 0$ and $x \oplus 0 = x$ for all x. In fact, $\oplus$ plays an important role in Einstein's theory of special relativity: It is the "correct" way to add two velocities, taking into account the relativistic notion that nothing can travel faster than light.

P-D. This question is readily solved by hand-counting once the value $\mathtt{C} = 25$ is known. This computation is made easier by employing symmetry. The $= 36$ possible terminal points are indicated as black dots in the figure below.



Note: One can also proceed analytically under the reasonable assumption that the total distance travelled is an integer. To this end, let $n = \sqrt{\texttt{C}}$ be the total distance travelled, and imagine the ant walking in the Cartesian plane, starting from $(0,0)$.

We'll first count all possible terminal points (x,y) with $x > 0$ and $y \geq 0$. (That is, points strictly within the first quadrant or on the positive x-axis.) By symmetry, the final count will be this 4 times this number, possibly plus one to account for the origin.

Observe that the ant can land on such a point (x,y) only if (1) both x and y are integers with $x+y \leq n$, and (2) the parity of $x+y$ matches the parity of n. The first condition is clear, and the second arises from the fact that the ant starts at $(0,0)$ and every one unit step it takes changes the parity of exactly one of its coordinates. These two conditions are neatly captured by the following single requirement:

$$x + y = n - 2k \quad \text{for some nonnegative integer } k.$$

To see that the ant can indeed land on any point (x,y) satisfying this requirement along with $x > 0$ and $y \geq 0$, simply observe that the following journey has exactly this outcome: First walk k units south, then x units east, and finally $k+y$ units north. The conditions $x > 0$ and $y \geq 0$ ensure that this walk does not retrace itself.

Let us now briefly consider the origin as a terminal point. Notice that this is possible if and only if $n = 2k$ for some $k \geq 2$, for in this case the ant can walk one unit east, $k-1$ units north, one unit west, and $k-1$ units south to arrive back at the origin. When n is odd or $n = 2$, the ant cannot finish his trip at the origin.

Finally, we must count integer pairs (x,y) satisfying $x+y = n-2k$, with $x > 0$, $y \geq 0$ and $k \geq 0$. For any fixed k, here are exactly $n-2k$ such pairs, namely $(1, n-2k-1), (2, n-2k-2), \ldots, (n-2k, 0)$. Summing over k then gives the tally

$$\begin{aligned}\sum_{k=0}^{\lfloor n/2 \rfloor} (n-2k) &= \begin{cases} n + (n-2) + (n-4) + \cdots + 4 + 2 + 0 & \text{if } n \text{ is even} \\ n + (n-2) + (n-4) + \cdots + 5 + 3 + 1 & \text{if } n \text{ is odd} \end{cases} \\ &= \begin{cases} \frac{n(n+2)}{4} & \text{if } n \text{ is even} \\ \frac{(n+1)^2}{4} & \text{if } n \text{ is odd.} \end{cases}\end{aligned}$$

Multiplying these expressions by 4 to account for symmetry and adding 1 in the even case to account for the origin shows that there are $(n+1)^2$ possible terminal points, for any $n > 2$.[1]

In the relay we had $n = 5$, which gives $6^2 = 36$ possible terminal points.

[1] When $n = 1$ the formula still works. When $n = 2$ we must subtract one to account for the origin.

Individual Relay Solutions

I-A. We have $S = 123 + 132 + 213 + 231 + 312 + 321 = 1332$, so $\texttt{A} = 1 + 3 + 3 + 2 = 9$.

Note: It is particularly easy to compute S, since upon writing the sum in columnar format, each of the 3 columns contains two copies of 1, 2, and 3. Thus the sum of every column is $2(1 + 2 + 3) = 12$, and $S = 100 \cdot 12 + 10 \cdot 12 + 1 \cdot 12 = 111 \cdot 12 = 1332$. This reasoning extends easily to calculate the sum of all permutations of any distinct digits. For instance, if we use the permute the digits $2, 3, 5$ and 7 in all possible ways and sum the resulting numbers, we obtain $(2 + 3 + 5 + 7) \cdot 3! \cdot 1111$.

I-B. A fraction is in lowest terms when its numerator and denominator have no common factors, so $\texttt{B}$ is the number of integers between 1 and 50 that have a common factor with $\texttt{A}$. Since $\texttt{A} = 9 = 3^2$, this is just the number of multiples of 3 between 1 and 50. There are $\texttt{B} = \left\lfloor \frac{50}{3} \right\rfloor = 16$ of these, namely $3, 6, 9, \ldots, 48$.

I-C. The unit cubes with two painted sides are precisely those that comprise the edges of the big cube, with the exception of the those at the corners of the big cube (which have three painted sides). Each of the 12 edges of the big cube contributes $\texttt{B} - 2$ such cubes, so $\texttt{C} = 12(\texttt{B} - 2)$. With $\texttt{B} = 16$ this gives $\texttt{C} = 168$.

I-D. Calculating the first few values of $f(n)$ leads to the following table:

n	1	2	3	4	5	6	7	8	9	10
$f(n)$	1	3	3	5	5	7	7	9	9	11

This data suggests that

$$f(n) = \begin{cases} n & \text{if } n \text{ is odd} \\ n+1 & \text{if } n \text{ is even.} \end{cases}$$

With $\texttt{C} = 168$, this gives $f(\texttt{C}) = 169$.

Note: The above formula for $f(n)$ is readily proved by mathematical induction.