Nova Scotia Math League

2014–2015

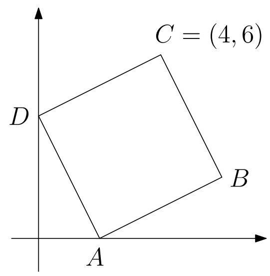
Game Two

Contest and Solutions

Team Questions

1. In how many distinct ways can you make change for \$10 using only quarters and dimes?

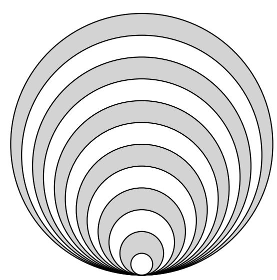
2. Square $ABCD$ has vertices A and D on the x- and y-axes, respectively, and $C = (4,6)$. Find the area of this square.



3. Find the units digit of
$$1 + 3 + 3^2 + 3^3 + 3^4 + \cdots + 3^{2015}.$$

4. An isosceles triangle has sides of lengths 18 and 41. Compute the area of the triangle.

5. In the diagram below, the twelve circles have diameters 1 through 12 and are mutually tangent. Find area of the shaded region.



6. Bob and Jim go for a run on a 400m circular track. They start at the same place and at the same time. Bob runs clockwise around the track at 13.4 km/h and Jim runs counterclockwise at 10.2 km/h. What is the straight-line distance between them after 15 minutes?

7. A coin is weighted so that heads are more likely than tails. When it is flipped twice there is a 48% chance of obtaining one head and one tail. What is the probability of heads on a single flip of the coin?

8. Let $f(x) = x^2 - 10x + 30$. Find all x such that $f(f(x)) = f(x)$.

9. The weather last June was very unpleasant. Every day was cloudy, rainy, or windy. On average, 5 out of 6 days were cloudy and 4 out of 5 were rainy or windy. It was never rainy without being cloudy, but 1/3 of the windy days weren't cloudy. It was rainy the same number of days as it was windy.

   On how many days was it both rainy and windy?

   Note: There are 30 days in June!

10. Each province and territory of Canada is to be coloured either red, green, or yellow in such a way that any two provinces/territories that share a border must have distinct colours. In how many ways can this be done?

Note:

- Regions that meet only at a "corner" are not considered to share a border. For instance, Saskatchewan is not adjacent to Nunavut.
- All land under the jurisdiction of any given province/territory must have the same colour. For instance, Newfoundland and Labrador must be similarly coloured, as do Baffin Island and Nunavut.

Team Question Answer Key

1. 21
2. 20
3. 0
4. 360
5. $\dfrac{39\pi}{2}$
6. $\dfrac{200\sqrt{2}}{\pi}$ metres
7. $\dfrac{3}{5}$
8. $4, 5, 6$
9. 6
10. 1728

Team Question Solutions

1. Certainly the number of quarters must be even and between 0 and 40. Note that there are 21 possibiities. And, given any such number of quarters, the balance of \$10 can be changed into dimes in exactly one way. So there are 21 possible ways to give change.

   Note: Algebraically, you are being asked to find nonnegative integer solutions (x, y) to the linear equation $25x + 10y = 1000$. When we restrict attention to *integer* solutions, we say that we are working with a *Diophantine equation*.

   The theory of *linear* Diophantine equations (such as $25x + 10y = 1000$ or $3x - 7y = 5$) is very straightforward and elegant. However, finding integer solutions to nonlinear equations (such as quadratics or cubics, etc.) is almost always very difficult. It is generally impossible even to determine when such solutions exist. The word "impossible" is used here with a precise technical meaning; essentially, we don't fail in our analysis of these equations because we're not smart enough, but rather because the task is beyond our current computational paradigm.

2. By symmetry $\triangle OAD$ is congruent to $\triangle PDC$. Thus $|OD| = |PC| = 4$ and $|OA| = |PD| = |OP| - |OD| = 6 - 4 = 2$. By Pythagorean Threorem, the area of the square is $|AD|^2 = |OA|^2 + |OD|^2 = 2^2 + 4^2 = 20$.

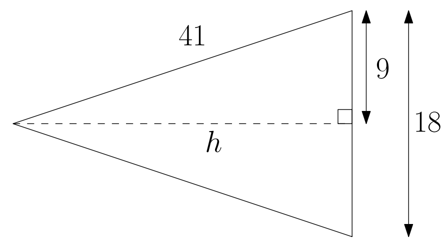
3. Some experimental computation shows the units digits of $3^0, 3^1, 3^2, 3^3, 3^4, 3^5, 3^6, \ldots$ to be $1, 3, 9, 7, 1, 3, 9, \ldots$. The pattern $1, 3, 9, 7$ repeats indefinitely, starting anew at each power 3^n where n is a multiple of 4. Since $2015 = 204 \cdot 4 - 1$, the units digit of $1 + 3 + 3^2 + \cdots + 3^{2015}$ is the same as the units digit of

   $$\overbrace{(1+3+9+7) + (1+3+9+7) + \cdots + (1+3+9+7)}^{\text{204 copies}} = 204 \cdot 20 = 4080,$$

   which is 0.

   Alternative Solution: Compute $1 + 3 + 3^2 + \cdots + 3^{2015} = \frac{1}{2}(3^{2016} - 1)$ by the usual formula for the sum of a geometric series. Observe that $3^{2016} \equiv (3^2)^{1008} \equiv (-1)^{1008} \equiv 1 \pmod 5$, while $3^{2016} \equiv (-1)^{2016} \equiv 1 \pmod 4$. Thus $3^{2016} - 1$ is divisible by 20, meaning $\frac{1}{2}(3^{2016} - 1)$ is divisible by 10 and consequently ends in a 0.

4. The triangle either has sides $(18, 18, 41)$ or $(41, 41, 18)$. The first case is impossible, since $18 + 18 < 41$, so the geometry is as indicated in the figure below. Pythagorean theorem gives $9^2 + h^2 = 41^2$, so $h^2 = 1600 \Longrightarrow h = 40$. Thus the area is $\frac{1}{2} \cdot 18 \cdot 40 = 360$.



5. The area of a circle with diameter d is $\pi d^2/4$, so the shaded area is

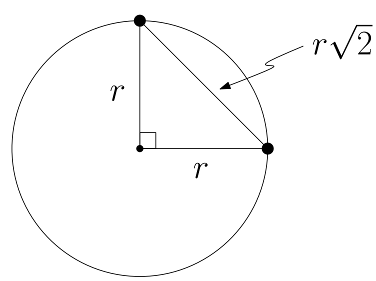
$$\frac{\pi}{4}\left(2^2-1^2+4^2-3^2+6^2-5^2+\cdots+12^2-11^2\right).$$

This is easy to compute upon observing that $(n+1)^2-n^2=2n+1$. Thus we have

$$\frac{\pi}{4}\left(3+7+11+15+19+23\right).$$

Now compute the sum directly or via the usual formula for arithmetic sequences to get $\frac{\pi}{4}\cdot 6\cdot(23+3)=39\pi/2$.

6. If we consider Jim's position as fixed, then Bob runs at a relative speed of $13.4+10.2=23.6$ km/h. In 15 mins, the relative distance covered is $\frac{1}{4}\cdot 23.6\cdot 1000=5900$ metres. Since $5900=400\cdot 14+300$, Bob and Jim will be 300m apart *along the track* in the clockwise direction, or equivalently 100m along the track in the counterclockwise direction. Since this is a quarter of the circular track, their straightline distance is $r\sqrt{2}$, where r is the radius of the track. (See the figure below.) The circumference of the track is 400 so its radius is $\frac{400}{2\pi}=\frac{200}{\pi}$. Hence Bob and Jim are $200\sqrt{2}/\pi$ metres apart after 15 minutes.



7. Let p be the probability of heads, so that $1-p$ is the probability of tails. The probability of both a heads and tails in two tosses of the coin is $p(1-p)+(1-p)p=2p(1-p)$. Setting this to equal $48\%=\frac{12}{25}$ leads to the quadratic $0=p^2-p+\frac{6}{25}=(p-\frac{2}{5})(p-\frac{3}{5})$. So either $p=\frac{3}{5}$ or $p=\frac{2}{5}$. Since we require $p>1-p$, we have $p=\frac{3}{5}$.

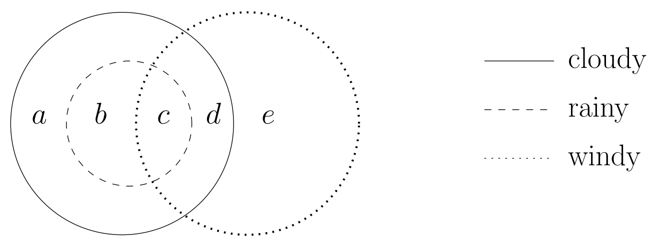
8. With $f(x) = x^2 - 10x + 30$, we have

$$\begin{aligned} f(f(x)) = f(x) &\iff f(x)^2 - 10f(x) + 30 = f(x) \\ &\iff f(x)^2 - 11f(x) + 30 = 0 \\ &\iff ((f(x) - 5)(f(x) - 6) = 0 \\ &\iff (x^2 - 10x + 25)(x^2 - 10x + 24) \\ &\iff (x-5)^2(x-4)(x-6) = 0. \end{aligned}$$

Therefore $x = 4, 5$ or 6.

Alternative Solution: First solve $f(y) = y$ to get $y = 5$ or $y = 6$, and then solve $f(x) = 5$ and $f(x) = 6$. This is simply a multi-step version of the above solution.

9. The Venn diagram below represents the interactions between the sets of cloudy, rainy, and windy days.



The given information translates into the following system of equations:

$$\begin{aligned} a + b + c + d + e &= 30 \\ a + b + c + d &= \tfrac{5}{6} \cdot 30 = 25 \\ b + c + d + e &= \tfrac{4}{5} \cdot 30 = 24 \\ \tfrac{1}{3}(c + d + e) &= e \\ b + c &= c + d + e. \end{aligned}$$

We wish to determine c. Subtracting the second equation from the first yields $e = 5$. Replacing $c + d + e$ with $b + c$ in the third and fourth equations (as per the fifth) then gives $\{2b + c = 24,\ b + c = 15\}$. This routinely gives $(b, c) = (9, 6)$. So there were $c = 6$ days that were both rainy and windy.

10. Begin by colouring the Yukon in any of 3 ways. Then BC can be coloured in either of the 2 remaining colours. Because NWT shares a border with both the Yukon and BC, the colour of the NWT is now forced (*i.e.* there is no choice). The colour of Alberta is also forced, since it borders NWT and BC, and the same is true of Saskatchewan, since it borders NWT and Alberta. Note that Nunavut and Manitoba, together, can now be

coloured in 3 distinct ways. (Why? Suppose NWT is red and Saskatchewan is green. Then Manitoba–Nunavut can be coloured green–red or green–yellow or yellow–red.) The remaining provinces are easy to colour: There are 2 choices for each of Ontario, Quebec, New Brunswick, Nova Scotia, and Newfoundland-Labrador, since when they are coloured in this order, each is adjacent to only one previously-coloured region. Finally, there are 3 ways to colour PEI. Thus the total number of colourings is

$$\overset{\text{Y}}{3} \cdot \overset{\text{BC}}{2} \cdot \overset{\text{M–N}}{3} \cdot \overset{\text{O}}{2} \cdot \overset{\text{Q}}{2} \cdot \overset{\text{NB}}{2} \cdot \overset{\text{NS}}{2} \cdot \overset{\text{NL}}{2} \cdot \overset{\text{PEI}}{3} = 1728.$$

Pairs Relay

P-A. Compute

$$\mathtt{A} = \frac{(4^2+4^2+4^2)(6^3+6^3+6^3)}{(2^4+2^4+2^4+2^4)(3^3+3^3+3^3+3^3+3^3+3^3)}.$$

$\boxed{\text{Pass on } \mathtt{A}}$

P-B. You will receive $\mathtt{A}$.

Let r and s be the roots of $x^2 - \mathtt{A}x + 1$, where $r < s$.

Let $\mathtt{B} = (r+1)(s+1)$. $\boxed{\text{Pass on } \mathtt{B}}$

P-C. You will receive $\mathtt{B}$.

Let $\mathtt{C}$ be the area of the triangle bounded between the graphs of $y = |x - \mathtt{B}|$ and $y = \mathtt{B}$.

$\boxed{\text{Pass on } \mathtt{C}}$

P-D. You will receive $\mathtt{C}$.

The sum of the **squares** of two consecutive **odd** positive integers is $\mathtt{C}(\mathtt{C}+1)$. Let $\mathtt{D}$ be the smaller of these integers.

$\boxed{\text{Done!}}$

Pairs Relay Answer Key

A. 3

B. 5

C. 25

D. 17

Pairs Relay Solutions

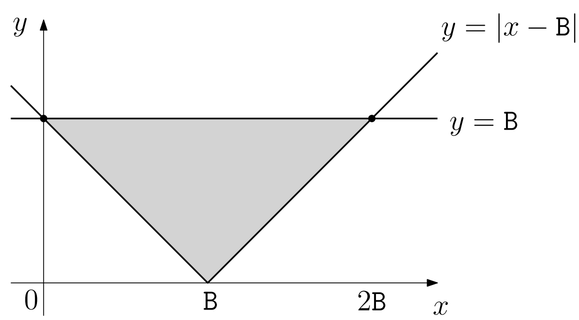
P-A. The quotient simplifies to

$$\mathtt{A} = \frac{(3\cdot 4^2)(3\cdot 6^3)}{(4\cdot 2^4)(6\cdot 3^3)} = \frac{(3\cdot 2^4)(3\cdot 2^3 3^3)}{(2^2\cdot 2^4)(2\cdot 3\cdot 3^3)} = \frac{2^7 3^5}{2^7 3^4} = 3.$$

P-B. We have $\mathtt{B} = rs + (r+s) + 1$. But the sum and product of the roots of $x^2 - \mathtt{A}x + 1$ are $\mathtt{A}$ and 1, respectively. Thus $\mathtt{B} = \mathtt{A} + 2$. With $\mathtt{A} = 3$ we get $\mathtt{B} = 5$.

Alternative Solution: The roots of $x^2 - \mathtt{A}x + 1$ are $\frac{1}{2}(\mathtt{A} \pm \sqrt{\mathtt{A}^2 - 4})$, by the quadratic formula. Then $\mathtt{A} = 3 \Longrightarrow r = \frac{1}{2}(3 - \sqrt{5}),\ s = \frac{1}{2}(3 + \sqrt{5}) \Longrightarrow (r+1)(s+1) = 5.$

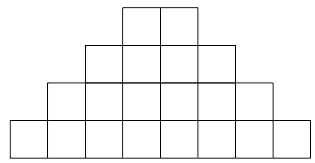
P-C. The bounded region is a triangle with base $2\mathtt{B}$ and height $\mathtt{B}$, as shown below. It therefore has area $\mathtt{C} = \frac{1}{2}\cdot 2\mathtt{B}\cdot \mathtt{B} = \mathtt{B}^2$. With $\mathtt{B} = 5$ we have $\mathtt{C} = 25$.



P-D. Let the consecutive odd integers be $\mathtt{D} = 2n - 1$ and $\mathtt{D} + 2 = 2n + 1$. We are told $(2n-1)^2 + (2n+1)^2 = \mathtt{C}(\mathtt{C}+1)$, which simplifies to $8n^2 + 2 = \mathtt{C}(\mathtt{C}+1)$. This gives $n = \sqrt{(\mathtt{C}+2)(\mathtt{C}-1)/8}$. With $\mathtt{C} = 25$ we have $n = \sqrt{27\cdot 3} = 9$ and hence $\mathtt{D} = 2\cdot 9 - 1 = 17$.

Individual Relay

I-A. A pyramid is formed by stacking unit squares as shown below. Let $\mathtt{A}$ be the perimeter of the pyramid that has 48 squares on its bottom row.



Pass on $\mathtt{A}$

I-B. You will receive $\mathtt{A}$.

Solve for $\mathtt{B}$:

$$\frac{1}{\frac{1}{\sqrt{\mathtt{A}}}+\frac{1}{\mathtt{B}}}=\frac{\sqrt{\mathtt{A}}}{3}.$$

Pass on $\mathtt{B}$

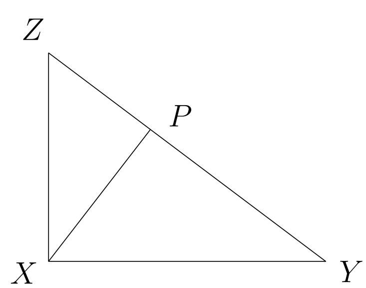
I-C. You will receive $\mathtt{B}$.

Today, Doug is twice as old as John. In $\mathtt{B}$ years, Doug will be twice as old as James. Let $\mathtt{C}$ be the age difference (in years) between John and James.

Pass on $\mathtt{C}$

I-D. You will receive $\mathtt{C}$.

Right triangle XYZ has legs $|XZ| = \mathtt{C}$ and $|XY| = \mathtt{C}+1$. Point P is on hypotenuse YZ such that $XP \perp YZ$. Find $|XP|$.



Done!

Individual Relay Answer Key

A. 144

B. 6

C. 3

D. $\dfrac{12}{5}$

Individual Relay Solutions

I-A. The pyramid with 48 squares on its bottom row has $48/2 = 24$ rows in total. Notice that the perimeter of this pyramid is the same as that of a 24×48 rectangle. Therefore $\mathtt{A} = 2(48 + 24) = 144$.

I-B. By cross multiplication we have $3 = 1 + \frac{\sqrt{\mathtt{A}}}{\mathtt{B}} \implies \mathtt{B} = \frac{\sqrt{\mathtt{A}}}{2}$. Then $\mathtt{A} = 144$ yields $\mathtt{B} = 6$.

I-C. Let d be Doug's age today. In $\mathtt{B}$ years, John's and James's ages will be $d/2 + \mathtt{B}$ and $(d + \mathtt{B})/2$, respectively, giving an age difference $\mathtt{C} = \frac{d}{2} + \mathtt{B} - \frac{d+\mathtt{B}}{2} = \frac{\mathtt{B}}{2}$. Setting $\mathtt{B} = 6$ gives $\mathtt{C} = 3$.

I-D. We must have $|XY| \cdot |XZ| = |ZY| \cdot |XP|$, since both the LHS and RHS equal twice the area of $\triangle XYZ$. We know $|XZ| = \mathtt{C}$ and $|XY| = \mathtt{C} + 1$, and Pythagorean theorem gives $|ZY| = \sqrt{|XZ|^2 + |XY|^2} = \sqrt{\mathtt{C}^2 + (\mathtt{C}+1)^2}$. Therefore

$$\mathtt{D} = |XP| = \frac{\mathtt{C}(\mathtt{C}+1)}{\sqrt{\mathtt{C}^2 + (\mathtt{C}+1)^2}}.$$

With $\mathtt{C} = 3$ we have $\mathtt{D} = \frac{12}{5}$.