Nova Scotia Math League

2014–2015

Game Three

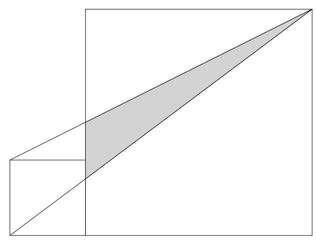
Contest and Solutions

Team Questions

1. Find the 2015^{th} digit after the decimal point when $\dfrac{1}{7}$ is written in decimal form.

2. The parabolas $y = x^2 - 3x + 2$ and $y = 3x^2 - 8x - 1$ intersect in two points. Find the x-intercept of the line passing through these points.

3. The figure below shows two adjacent squares, with lines drawn from two corners of the small square to a corner of the large square. The small square has area 16 and the large square has area 144. Find the area of the shaded triangle.



4. A rectangular field measures 300m by 600m. A fence is erected around the field, with posts spaced 10m apart along the perimeter (including one post in each corner).

   Unfortunately, the posts *should* have been installed 8m apart on the long sides of the field and 15m apart on the short sides.

   What is the minimum number of posts that will have to be moved to correct this error?

5. Bob throws a fair 6-sided die three times in a row. Find the probability that his highest roll is a 5.

6. How many integers between 1 and 30 (inclusive) can be expressed as a sum of two or more consecutive positive integers?

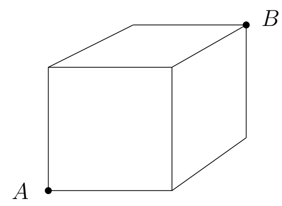
   For instance, 12 is such a number since $12 = 3 + 4 + 5$.

7. When 1 litre of water is poured into a circular cone it reaches a depth of 5cm. What is the depth when an additional litre of water is poured into the cone?

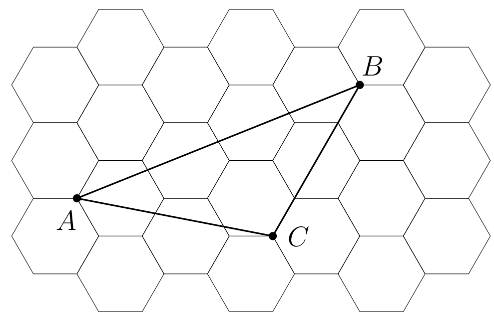
8. Evaluate the following product:

$$\frac{4}{3} \cdot \frac{9}{8} \cdot \frac{16}{15} \cdot \frac{25}{24} \cdot \cdots \cdot \frac{2015^2}{2015^2 - 1}$$

9. An ant walks from corner A of a cube to the diagonally opposite corner B. If the ant walks only along edges and never hits the same point twice, how many distinct paths can it take?



10. The side of each regular hexagon in the honeycomb lattice below is of length 1. Find the perimeter of $\triangle ABC$.



Team Question Answer Key

1. 5
2. 7 or $(7,0)$
3. 18
4. 130
5. $\dfrac{61}{216}$
6. 25
7. $5\sqrt[3]{2}$
8. $\dfrac{2015}{1008}$
9. 18
10. $11+\sqrt{21}$

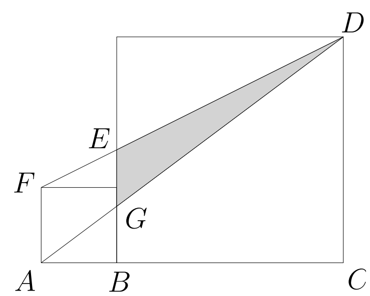
Team Question Solutions

1. Dividing 7 into 1 gives $\frac{1}{7} = 0.\overline{142857}$. The length of the period is 6, and since $2015 = 335 \cdot 6 + 5$, the 2015th digit after the decimal point will be the same as the 5th digit after the decimal point, namely 5.

2. Notice that if (x, y) lies on both parabolas, then $y = x^2 - 3x + 2$ and $y = 3x^2 - 8x - 1$, and taking the second equation minus 3 times the first gives $-2y = x - 7$. So if A and B are the points of intersection of the parabolas, then A and B lie on the line $-2y = x - 7$. Set $y = 0$ to see that the line crosses the x-axis at $x = 7$.

   Alternative Solution: Set $x^2 - 3x + 2 = 3x^2 - 8x - 1$ to get $0 = 2x^2 - 5x - 3 = (2x + 1)(x - 3)$, which gives $x = -\frac{1}{2}$ or $x = 3$. Plug these values back into $y = x^2 - 3x + 2$ to get intersection points $(-\frac{1}{2}, \frac{15}{4})$ and $(3, 2)$, and find the equation of the line between these two points as usual.

3. With the diagram labelled as below, observe $|AB| = \sqrt{16} = 4$, $|BC| = \sqrt{144} = 12$, and $\triangle GDE$ is similar to $\triangle ADF$.

   The heights of $\triangle GDE$ and $\triangle ADF$ are in ratio $|BC| : (|AB| + |BC|) = 12 : 16 = 3 : 4$, so their areas are in ratio $9 : 16$. The area of $\triangle ADF$ is $\frac{1}{2}|AF| \cdot |AC| = \frac{1}{2} \cdot 4 \cdot (4 + 12) = 32$. Therefore the area of $\triangle GDE$ is $\frac{9}{16} \cdot 32 = 18$.

4. Each long side of the field originally has $600/10 = 60$ posts while each short side has $300/10 = 30$. (We associate one corner post with each side.) Of the 60 posts on each long side, $600/\text{lcm}(10, 8) = 600/40 = 15$ will *not* have to be moved; and of the 30 posts on each short side, $300/\text{lcm}(10, 15) = 300/30 = 10$ will remain where they are. So a total of $2(60 - 15) + 2(30 - 10) = 130$ posts will have to be moved.

5. There are 5^3 ways in which Bob can score at most five on each throw. Of these possibilities, there are 4^3 in which Bob scores at most four on each throw. So there are $5^3 - 4^3 = 61$ ways in which Bob's highest roll will be a five. The required probability is therefore $61/6^3 = \frac{61}{216}$.

Alternative Solution: Bob can either roll 1, 2, or 3 fives. There are $3 \cdot (1 \cdot 4 \cdot 4) = 48$ ways of rolling a single five: it can come on the first, second, or third roll; and the other rolls can be 1,2,3, or 4. There are $3 \cdot (1 \cdot 1 \cdot 4) = 12$ ways of rolling two fives: the non-five roll can come first, second, or third, and it can be 1,2,3, or 4. Finally, there is only 1 way in which he can roll 3 fives. So the desired probability is $(48 + 12 + 1)/216 = \frac{61}{216}$.

Note: Then the first solution above applies in the more general setting where Bob throws the die n times. In this case, the probability his highest roll is five is $(5^n - 4^n)/6^n$. It's less clear how the second solution can be modified, but it is possible to do so. They key turns out to be the binomial theorem, since the binomial coefficient $\binom{n}{k}$ counts the number of ways that k fives can be distributed amongst the n rolls.

6. Some experimentation leads us to believe that $1, 2, 4, 8$ and 16 are the only numbers between 1 and 30 that are *not* representable as a sum of consecutive positive integers. This is indeed the case, and of course it is no coincidence that each of these numbers is a power of 2.

   The sum of the consecutive integers between m and n, inclusive, is

$$\begin{aligned} m + (m+1) + \cdots + n &= (1 + 2 + 3 + \cdots + n) - (1 + 2 + 3 + \cdots + m) \\ &= \frac{n(n+1)}{2} - \frac{m(m+1)}{2} \\ &= \frac{1}{2}(n-m)(n+m+1). \end{aligned}$$

   Notice that one of $n - m$ or $n + m + 1$ must be even and the other odd. In particular, the above sum must have an odd divisor, and can therefore not be a power of 2.

   Conversely, if N is *not* a power of 2, then it must have an odd divisor. Say $N = (2k+1)r$ for some integers k, r. If $k \geq r$, then N is the sum of the $2r$ consecutive integers between $k - r + 1$ and $k + r$. If $r > k$, then N is the sum of the $2k + 1$ consecutive integers from $r - k$ to $r + k$. (These cases amount to letting $(m, n) = (k - r + 1, k + r)$ and $(m, n) = (r - k, r + k)$, above.)

   So we have arrived at an elegant result: A number is expressible as a sum of two or more consecutive positive integers if and only if it is *not* a power of 2.

7. The volume must scale cubically with respect to depth (a linear measure). Since an additional 1 litre of water doubles the volume, the depth must scale by $\sqrt[3]{2}$. Therefore the resulting depth will be $5\sqrt[3]{2}$.

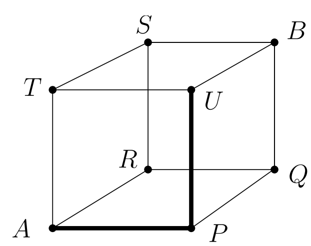
   Alternative Solution: A circular cone with base radius r and height h has volume $\frac{1}{3}\pi r^2 h$. So we originally have $1000 = \frac{5}{3}\pi r^2$ (since 1 litre = 1000 cubic centimetres), and an additional litre of water gives $2000 = \frac{1}{3}\pi R^2 h$, where H is the new depth and R the new base radius. Since $R : r = H : 5$, dividing the second equation by the first results in $2 = (h/5)^3$, whence $h = 5\sqrt[3]{2}$.

8. Using the difference of squares factorization $n^2 - 1 = (n-1)(n+1)$, we have

$$\begin{aligned} &\frac{2^2}{2^2-1} \cdot \frac{3^2}{3^2-1} \cdot \frac{4^2}{4^2-1} \cdot \cdots \cdot \frac{2015^2}{2015^2-1} \\ &= \frac{2\cdot 2}{1\cdot 3} \cdot \frac{3\cdot 3}{2\cdot 4} \cdot \frac{4\cdot 4}{3\cdot 5} \cdot \frac{5\cdot 5}{4\cdot 6} \cdot \frac{6\cdot 6}{5\cdot 7} \cdot \cdots \cdot \frac{2014\cdot 2014}{2013\cdot 2015} \cdot \frac{2015\cdot 2015}{2014\cdot 2016} \\ &= \frac{2\cdot 2015}{1\cdot 2016} \\ &= \frac{2015}{1008}, \end{aligned}$$

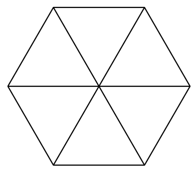
where the simplification comes from cancelling equal factors in the numerator and denominator.

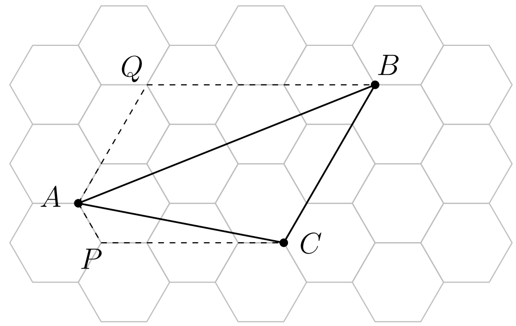
9. Label the points of the cube as indicated below. Each path from A to B must begin with a walk from A to one of P, T, or S. By symmetry, there are the same number of paths that begin in each manner, so we focus on those that begin with a walk from A to P. Any such path must proceed from P to either U or Q, and again by symmetry there will be the same number of paths of each type. So the number of distinct paths from A to B will be $3 \cdot 2 = 6$ times the number of paths that begin with the steps $A \to P \to U$. (These initial steps have been highlighted in the diagram.)



From U the ant can proceed to B via a unique path of length 1 ($U \to B$); or a unique path of length 3 ($U \to T \to S \to B$); or a unique path of length 5 ($U \to T \to S \to R \to Q \to B$). So there are a total of $3 \cdot 2 \cdot 3 = 18$ distinct paths from A to B.

10. Draw points P and Q as indicated in the diagram below and consider triangles $\triangle ABQ$ and $\triangle APC$. Dissecting each hexagon into 6 equilateral triangles shows their diameter to be 2. So $|BC| = 1 + 2 + 1 = 4$, and similarly $|PC| = 4$, $|AQ| = 3$, and $|QB| = 5$.





Clearly $\angle APC = \angle BQA = 120^\circ$. But if a triangle has sides of lengths a and b containing a 120° angle, then cosine law shows the third side to be of length $\sqrt{a^2 + b^2 + ab}$. Therefore

$$\begin{aligned}
|AC| &= \sqrt{|AP|^2 + |PC|^2 + |AP||PC|} \\
&= \sqrt{1^2 + 4^2 + 1 \cdot 4} \\
&= \sqrt{21} \\
|AB| &= \sqrt{|AQ|^2 + |QB|^2 + |AQ||QB|} \\
&= \sqrt{3^2 + 5^2 + 3 \cdot 5} \\
&= 7,
\end{aligned}$$

and the perimeter of $\triangle ABC$ is $4 + 7 + \sqrt{21} = 11 + \sqrt{21}$.

Pairs Relay

P-A. Let $\mathsf{A} = \left(\sqrt{2+\sqrt{3}} + \sqrt{2-\sqrt{3}}\right)^2$.

$\boxed{\text{Pass on A}}$

P-B. You will receive A.

The lines $y = \mathsf{A}x + \mathsf{B}$, $y = 2\mathsf{A}x + 2$ and $y = 2x + 2\mathsf{A}$ all pass through the same point.

$\boxed{\text{Pass on B}}$

P-C. You will receive B.

Real numbers x and y satisfy $\dfrac{3x+2y}{2x-3y} = \mathsf{B}$.

Let $\mathsf{C} = \dfrac{x^2}{y^2}$.

$\boxed{\text{Pass on C}}$

P-D. You will receive C.

A party is attended by C couples. Each person shakes hands with everyone except their partner (and themselves!)

Let D be the total number of handshakes.

$\boxed{\text{Done!}}$

Pairs Relay Answer Key

A. 6

B. 8

C. 4

D. 24

Pairs Relay Solutions

P-A. Use $(a+b)^2 = a^2 + 2ab + b^2$ to compute

$$\begin{aligned} \mathsf{A} = \left(\sqrt{2+\sqrt{3}} + \sqrt{2-\sqrt{3}}\right)^2 &= 2+\sqrt{3} + 2\sqrt{(2+\sqrt{3})(2-\sqrt{3})} + 2 - \sqrt{3} \\ &= 4 + 2\sqrt{2^2 - 3} \\ &= 6. \end{aligned}$$

P-B. First find the intersection of lines $y = 2\mathsf{A}x + 2$ and $y = 2x + 2\mathsf{A}$ by setting $2\mathsf{A}x + 2 = 2x + 2\mathsf{A}$. Either $\mathsf{A} = 1$, in which case the lines are identical, or we find the unique intersection point $(x, y) = (1, 2\mathsf{A} + 2)$. In the former case there would not be a unique value of B for which the third line $y = \mathsf{A}x + \mathsf{B}$ meets the others at a point (any value of B would suffice). So we can safely assume that all lines pass through $(1, 2\mathsf{A} + 2)$. But this point lies on $y = \mathsf{A}x + \mathsf{B}$ if and only if $\mathsf{A} + \mathsf{B} = 2\mathsf{A} + 2$, or simply $\mathsf{B} = \mathsf{A} + 2$. With $\mathsf{A} = 6$ we get $\mathsf{B} = 8$.

P-C. Cross multiply and rearrange to get

$$\begin{aligned} 3x + 2y = \mathsf{B}(2x - 3y) &\Longrightarrow (2\mathsf{B} - 3)x = (2 + 3\mathsf{B})y \\ &\Longrightarrow \frac{x}{y} = \frac{2 + 3\mathsf{B}}{2\mathsf{B} - 3} \\ &\Longrightarrow \frac{x^2}{y^2} = \left(\frac{2 + 3\mathsf{B}}{2\mathsf{B} - 3}\right)^2. \end{aligned}$$

With $\mathsf{B} = 8$ we get $\mathsf{C} = x^2/y^2 = 2^2 = 4$.

P-D. There are $2\mathsf{C}$ people at the party. Each person shakes hands with $2\mathsf{C} - 2$ others, so there are a total of $\mathsf{D} = \frac{1}{2} \cdot 2\mathsf{C}(2\mathsf{C} - 2) = \mathsf{C}(2\mathsf{C} - 2)$ handshakes. (Note that we divide by 2 because Alan shaking hands with Bob is the same as Bob shaking hands with Alan.) With $\mathsf{C} = 4$ we get $\mathsf{D} = 24$.

Individual Relay

I-A. Reducing two sides of an $\mathsf{A} \times \mathsf{A}$ square by 60% and enlarging the other two by 60% results in a rectangle with an area of 64 square units.

$\boxed{\text{Pass on } \mathsf{A}}$

I-B. You will receive A.

Let B be the smallest positive integer such that $20\mathsf{B} + \mathsf{A}$ is divisible by 7.

$\boxed{\text{Pass on } \mathsf{B}}$

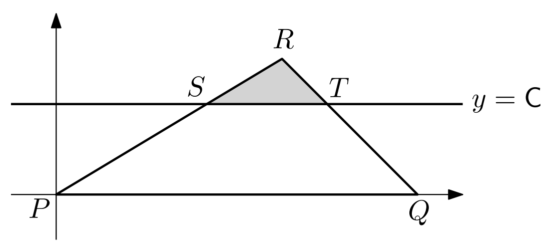
I-C. You will receive B.

For any nonzero x and y, let $x \oplus y = \dfrac{x+y}{xy}$.

Compute $\mathsf{C} = (\mathsf{B} \oplus 1) \oplus (1 \oplus \mathsf{B})$. $\boxed{\text{Pass on } \mathsf{C}}$

I-D. You will receive C.

Triangle PQR has $P = (0,0)$, $Q = (8,0)$ and $R = (5,3)$. The line $y = \mathsf{C}$ intersects PR and QR at S and T, respectively.



Let D be the area of $\triangle RST$. $\boxed{\text{Done!}}$

Individual Relay Answer Key

A. 10

B. 3

C. $\dfrac{3}{2}$

D. 3

Individual Relay Solutions

I-A. The rectangle has dimensions $0.4\mathsf{A} \times 1.6\mathsf{A}$ and therefore has area $\frac{2}{5} \cdot \frac{8}{5}\mathsf{A}^2 = \frac{16}{25}\mathsf{A}^2$. Solving $64 = \frac{16}{25}\mathsf{A}^2$ yields $\mathsf{A} = 10$.

I-B. Since $20\mathsf{B} + \mathsf{A} = 21\mathsf{B} + (\mathsf{A} - \mathsf{B})$, we see that $20\mathsf{B} + \mathsf{A}$ will be divisible by 7 if and only if $\mathsf{A} - \mathsf{B}$ is divisible by 7. So the desired value of B is simply the remainder when A is divided by 7. With $\mathsf{A} = 10$ we have $\mathsf{B} = 3$.

Note: In the language of modular arithmetic, we have $20\mathsf{B} + \mathsf{A} \equiv -\mathsf{B} + \mathsf{A} \pmod 7$. So $20\mathsf{B} + \mathsf{A}$ is divisible by $7 \iff -\mathsf{B} + \mathsf{A} \equiv 0 \pmod 7 \iff \mathsf{B} \equiv \mathsf{A} \pmod 7$.

I-C. For any $x \neq 0$ we have $x \oplus x = 2x/x^2 = 2/x$. Since $\mathsf{B} \oplus 1 = 1 \oplus \mathsf{B} = \frac{1+\mathsf{B}}{\mathsf{B}}$, it follows that

$$\mathsf{C} = (\mathsf{B} \oplus 1) \oplus (1 \oplus \mathsf{B}) = \frac{2}{\frac{1+\mathsf{B}}{\mathsf{B}}} = \frac{2\mathsf{B}}{1+\mathsf{B}}.$$

With $\mathsf{B} = 3$ we have $\mathsf{C} = \frac{3}{2}$.

I-D. Note that $\triangle RST$ is similar to $\triangle RPQ$. The height (from R) of $\triangle RST$ is $3 - \mathsf{C}$, whereas the height of $\triangle RPQ$ is 3. Therefore

$$\begin{aligned} \mathsf{D} = \text{Area}(\triangle RST) &= \left(\frac{3-\mathsf{C}}{3}\right)^2 \text{Area}(\triangle RPQ) \\ &= \left(\frac{3-\mathsf{C}}{3}\right)^2 \cdot \frac{8 \cdot 3}{2} \\ &= \frac{4(3-\mathsf{C})^2}{3}. \end{aligned}$$

With $\mathsf{C} = \frac{3}{2}$ we have $\mathsf{D} = 3$.

Note: Once $\mathsf{C} = \frac{3}{2}$ is known, it is immediate that the area of $\triangle RST$ one quarter that of $\triangle RPQ$, since the former triangle is clearly half the size of the latter.