Nova Scotia Math League

2015–2016

Game One

PROBLEMS AND SOLUTIONS

Team Questions

1. All of Bill's math tests are graded out of 100 points. If Bill scores 95% on his next test, then his average for the year increases by 3 points. If he scores 75%, then his average decreases by 1 point.

   How many math tests has Bill written so far this year?

2. Jane and Tracy begin running around a 300m oval track. They start from the same spot but head out in opposite directions, with Jane running at 6 m/s and Tracy jogging at half that speed.

   How far (in metres) has Jane run when she meets Tracy for the second time?

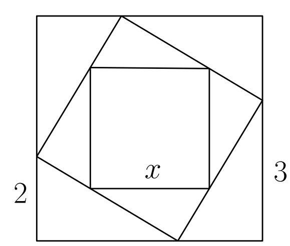
   Note: Their starting point is not considered their first meeting!

3. Find n such that
$$\frac{1+3+5+7+\cdots+(2n-1)}{2+4+6+8+\cdots+2n} = \frac{2015}{2016}.$$

4. Let $N = \overbrace{11111\cdots1111}^{1000\text{ ones}}$ and let $M = 2015N$.

   Calculate the sum of the digits of M.

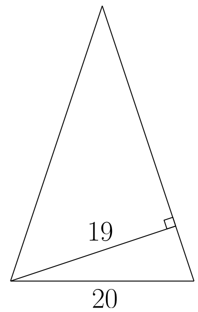
5. The figure below shows three squares nested within each other, with the sides of the smallest and largest squares parallel. Find x, the length of each side of the smallest square.



6. The function f satisfies $f(x) = 3f(1-x) + x^3$ for all real numbers x.

   Determine $f(3)$.

7. An isosceles triangle has base 20 and the altitude to one of its equal sides is 19. Find the length of the two equal sides.



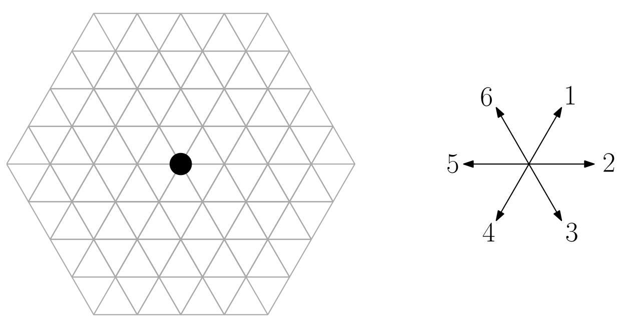
8. Determine the sum of the cubes of the three roots of $x^3 - 2x + 1 = 0$.

9. A magician lays 5 playing cards on a table. He asks you to select any number of the cards (possibly none at all). He removes your selected cards from the table and then asks you to flip over any number of those remaining.

   In how many ways can you perform the magician's tasks?

   Note: The order in which you flip over the cards does not matter.

10. A bead is placed at a vertex of a hexagonal lattice. It is pushed around the lattice, traversing one edge at a time, with the direction of each move randomly determined by the throw of a 6-sided die (see diagram).



   Find the probability that the bead returns to its starting position after 4 moves.

Team Question Solutions

1. A 20 point difference (75 to 95) on the next test amounts to 4 point difference in overall average. So if the next test is Bill's $(n+1)-$st, then $20/(n+1) = 4$. Thus $n = 4$.

2. Since Jane runs twice as fast as Tracy, the distances they cover in a given period of time are in the ratio 2 : 1. Therefore they meet after every $\frac{2}{3} \cdot 300 = 200$ m Jane runs. So Jane has run 400 m when they meet for the second time.

3. Some experimentation reveals

$$\begin{aligned} 1+3 &= 2^2 \\ 1+3+5 &= 3^2 \\ 1+3+5+7 &= 4^2, \end{aligned}$$

which suggests the identity

$$1+3+5+7+\cdots+(2n-1) = n^2.$$

This is indeed true, and is easy to prove. Thus we have

$$\begin{aligned} \frac{2015}{2016} &= \frac{1+3+5+\cdots+(2n-1)}{2+4+6+\cdots+2n} \\ &= \frac{n^2}{2(1+2+3+\cdots+n)} \\ &= \frac{n^2}{2\cdot\frac{1}{2}n(n+1)} \\ &= \frac{n}{n+1}, \end{aligned}$$

from which it is evident that $n = 2015$.

Note: It is well known that the sum of the first n natural numbers is

$$1+2+3+\cdots+n = \frac{1}{2}n(n+1).$$

This is readily proved by grouping the terms in pairs that sum to $n+1$ (e.g. first and last, second and second-last, etc.) From this, the sum of any arithmetic sequence is readily derived. For instance, the first n terms of the sequence $a, a+d, a+2d, a+3d, \ldots$ is given by

$$\frac{n}{2}\left(2a+(n-1)d\right).$$

Setting $a = 1$ and $d = 2$ gives $1+3+5+\cdots+(2n-1) = n^2$.

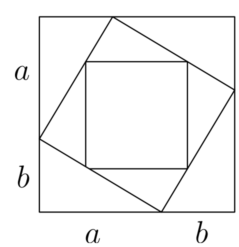
4. Consider performing "long multiplication" to compute the product $2015N$ in the more manageable case $N = 111111$. Here we have:

$$\begin{array}{r}
2\,0\,1\,5 \\
\times\;\; 1\,1\,1\,1\,1\,1 \\
\hline
2\,0\,1\,5 \\
2\,0\,1\,5 \\
2\,0\,1\,5 \\
2\,0\,1\,5 \\
2\,0\,1\,5 \\
2\,0\,1\,5 \\
\hline
2\,2\,3\,8\,8\,8\,6\,6\,5
\end{array}$$

The digits of the product are computed by adding columns of digits, as usual. But since no carries occur when doing so, the sum of the digits in the product is simply the sum of all digits occurring in the intermediary steps, namely $6 \cdot (2 + 0 + 1 + 5)$.

The same logic applies when N is composed of 1000 ones. In this case, the desired sum of digits is instead $1000 \cdot (2 + 0 + 1 + 5) = 8000$.

5. The general situation is illustrated below. The middle square divides the sides of the large square in the ratio $a : b$ and. Since the sides of the small square are parallel to those of the large square, the sides of the middle square are also divided in ratio $a : b$.



The side of the larger square is $a + b$ while that of the middle is (by Pythagorean Theorem) $\sqrt{a^2 + b^2}$, giving a "compression" factor of

$$\frac{\sqrt{a^2 + b^2}}{a + b}.$$

In passing from the middle to the small square, lengths will be compressed by this same factor. Thus the side of the small square is

$$\sqrt{a^2 + b^2} \cdot \frac{\sqrt{a^2 + b^2}}{a + b} = \frac{a^2 + b^2}{a + b}.$$

Setting $a = 3$ and $b = 2$ gives $\frac{13}{5}$.

Note: One can readily complete the problem without considering the general situation. However, the solution given above answers the question for any number of nested squares, rather than just three.

6. Setting $x = 3$ and $x = -2$ in the identity $f(x) = 3f(1-x) + x^3$ gives the simultaneous equations

$$\begin{aligned} f(3) &= 3f(-2) + 27 \\ f(-2) &= 3f(3) - 8. \end{aligned}$$

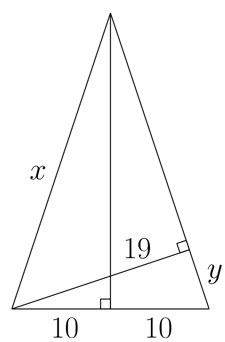
These are readily solved to give $f(3) = -\frac{3}{8}$.

Note: More generally, setting $x = t$ and $x = 1-t$ in $f(x) = 3f(1-x) + x^3$ gives

$$\begin{aligned} f(t) &= 3f(1-t) + t^3 \\ f(1-t) &= 3f(t) + (1-t)^3. \end{aligned}$$

Solving for $f(t)$ identifies the mysterious function as $f(t) = \frac{1}{8}(2t^3 - 9t^2 + 9t - 3)$.

7. Let x be the length of the equal sides. Drop a perpendicular to the base and let y be as indicated in the diagram below.



Then Pythagorean theorem gives $y = \sqrt{20^2 - 19^2} = \sqrt{39}$, and by similar triangles we have $\frac{x}{10} = \frac{20}{y}$. Therefore $x = 200/\sqrt{39}$.

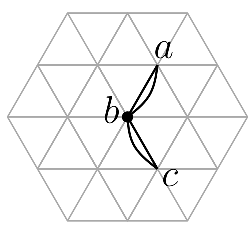
8. Let the roots be a, b and c. Then

$$\begin{aligned} a^3 - 2a + 1 &= 0 \\ b^3 - 2b + 1 &= 0 \\ c^3 - 2c + 1 &= 0. \end{aligned}$$

Summing and rearranging gives $a^3 + b^3 + c^3 = 2(a+b+c) - 3$. But $a+b+c$ is the sum of the roots of $x^3 - 2x + 1$, which we know to be the negative of the coefficient of x^2, namely 0. So $a^3 + b^3 + c^3 = 2 \cdot 0 - 3 = -3$.

9. You have 3 choices for each card: (1) remove it from the table, (2) keep it but don't flip it, and (3) keep it and flip it. Since there are 5 cards, there are $3^5 = 243$ possibilities altogether.

10. There are 3 classes of path the bead can take to return to its starting position after four moves. These are depicted below, where labels a, b and c indicate the vertices that the bead occupies after its 1st, 2nd, and 3rd moves.



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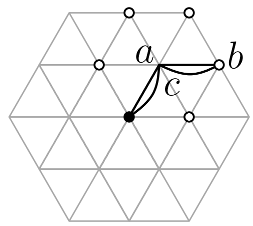

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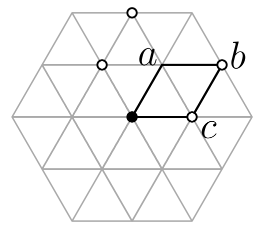

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We can count these individually, as follows:

(1) There are 6 choices for each of a and c, giving $6 \cdot 6$ paths.

(2) For each of the 6 choices for a $(= c)$ there are 5 choices for b (indicated by open circles) giving $6 \cdot 5$ paths of this type.

(3) For each of the 6 choices for a there are 4 choices for bs (indicated by open circles), and these choices determine c. There are $6 \cdot 4$ paths of this type.

Thus the probability of the bead returning to its starting position after 4 moves is

$$\frac{6 \cdot 6 + 6 \cdot 5 + 6 \cdot 4}{6^4} = \frac{6 + 5 + 4}{6^3} = \frac{15}{216} = \frac{5}{72}.$$

Pairs Relay

P-A. The faces of a strange 6-sided die are numbered 1, 1, 2, 3, 5, 8. The die is rolled twice and the results are summed.

Let A be the smallest integer greater than 1 that is **not** a possible sum.

$\boxed{\text{Pass on A}}$

P-B. You will receive A.

The difference and the quotient of two numbers both equal A.

Let B be the **smaller** number. $\boxed{\text{Pass on B}}$

P-C. You will receive B. Suppose $\mathsf{B} = p/q$, expressed in lowest terms.

The lines and $y = 3x - p$ and $y = -x + q$ intersect at a point P. Let (m, n) be the point nearest to P that has integer coordinates.

Let $\mathsf{C} = mn$. $\boxed{\text{Pass on C}}$

P-D. You will receive C.

Donuts cost $1 each, but for every 3 you purchase you get one free. Suppose you leave the store with C donuts.

Let D be the number of dollars you spent.

$\boxed{\text{Done!}}$

Pairs Relay Solutions

P-A. It is straightforward to verify that all sums between 2 and 11, inclusive, are achievable, while 12 is not. So $\mathsf{A} = 12$.

P-B. We assume $\mathsf{A} > 1$, knowing similar logic will apply if not. Let the larger number be x and the smaller B. Then we have

$$x - \mathsf{B} = \mathsf{A} \qquad \text{and} \qquad \frac{x}{\mathsf{B}} = \mathsf{A}.$$

The second equation gives $x = \mathsf{AB}$, and substitution into the first yields

$$\mathsf{B} = \frac{\mathsf{A}}{\mathsf{A} - 1}.$$

With $\mathsf{A} = 12$ we get $\mathsf{B} = \frac{12}{11}$.

P-C. We have the system $\{y = 3x - p, y = -x + q\}$. Subtract the second equation from the first to get $x = (p + q)/4$, and substitute into $y = -x + q$ to get $y = (3q - p)/4$. So the intersection point of the two lines is

$$\left(\frac{p+q}{4}, \frac{3q-p}{4}\right).$$

From $\mathsf{B} = \frac{12}{11}$ we obtain $p = 12$ and $q = 11$, giving the intersection point $(\frac{23}{4}, \frac{21}{4})$. The nearest integer point is therefore $(m, n) = (6, 5)$. Therefore $\mathsf{C} = mn = 30$.

P-D. If you spend \$D then you get $\mathsf{D} + \left\lfloor \frac{\mathsf{D}}{3} \right\rfloor$ donuts, where $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x. Thus $\mathsf{D} + \left\lfloor \frac{\mathsf{D}}{3} \right\rfloor = \mathsf{C}$.

Since $\left\lfloor \frac{x}{3} \right\rfloor \approx \frac{x}{3}$, we get $\mathsf{D} + \frac{\mathsf{D}}{3} \approx \mathsf{C}$, or simply $\mathsf{D} \approx 3\mathsf{C}/4$.

When $\mathsf{C} = 30$ we have $\mathsf{D} \approx 22.5$. Evaluating $\mathsf{D} + \left\lfloor \frac{\mathsf{D}}{3} \right\rfloor$ at $\mathsf{D} = 22$ and $\mathsf{D} = 23$ shows $\mathsf{D} = 23$ to be the correct answer.

Note: The equation $\mathsf{D} + \left\lfloor \frac{\mathsf{D}}{3} \right\rfloor = 30$ can be solved analytically by considering the remainder when D is divided by 3. But it is very efficient to first approximate and then verify the solution, as above.

Individual Relay

I-A. My two sons share the same birthday, but they were born three years apart. Their ages are now 1/3 and 1/4 of mine.

Let A be my age. $\boxed{\text{Pass on A}}$

I-B. You will receive A.

If I received $\sqrt{\mathsf{A}}$ more points on my test, my grade would increase from 76% to 84%.

Let B be the total number of points on the test. $\boxed{\text{Pass on B}}$

I-C. You will receive B.

Let C be the smallest possible value of $(p - \mathsf{B})^2$, where p is a prime number.

$\boxed{\text{Pass on C}}$

I-D. You will receive C.

Let D be the maximum value that can be obtained by inserting a single pair of parentheses (brackets) into the expression

$$10 + 3 \times \mathsf{C} + 14$$

.

$\boxed{\text{Done!}}$

Individual Relay Solutions

I-A. From the given information we have $\frac{\mathsf{A}}{3} - \frac{\mathsf{A}}{4} = 3$, which yields $\mathsf{A} = 36$.

I-B. We are told that $\sqrt{\mathsf{A}}$ additional points increases our grade by 8%. Therefore $\sqrt{\mathsf{A}}/\mathsf{B} = 8/100$. With $\mathsf{A} = 36$ this gives $\mathsf{B} = 75$.

I-C. Clearly $(p - \mathsf{B})^2$ will be minimized when p is either the largest prime less than B or the smallest prime larger than B. With $\mathsf{B} = 75$ these primes are 73 and 79, respectively, and we see that $(p - \mathsf{B})^2$ is has smallest value $2^2 = 4$ when $p = 73$. Therefore $\mathsf{C} = 4$.

I-D. Inserting a single pair of parenthesis results in one of the following values for the expression:

$$10 + 3 \times \mathsf{C} + 14 = 3\mathsf{C} + 24$$
$$(10 + 3) \times \mathsf{C} + 14 = 13\mathsf{C} + 14$$
$$10 + 3 \times (\mathsf{C} + 14) = 3\mathsf{C} + 52.$$

The first expression is clearly less than the third, so only the 2nd and 3rd need be considered. The 2nd is the largest when $13\mathsf{C} + 14 \geq 3\mathsf{C} + 52 \iff \mathsf{C} \geq \frac{19}{5}$, and otherwise the 3rd is biggest.

With $\mathsf{C} = 4$ we find the 2nd expression is the largest, having value $\mathsf{D} = 66$.