A Probability Work Sheet

October 19, 2006

Introduction: Rolling a Die

Suppose Geoff is given a fair six-sided die, which he rolls. What are the chances he rolls a six? In order to solve this problem, we need to understand the concept of **probability**.

The Definitions

The **probability of an event** is a number $\rho \in [0, 1]$ which tells us the likelihood of the event occurring. The closer ρ is to one, the more likely the event will occur. The closer ρ is to zero, the less likely the event will occur. In order to calculate the probability, we need the following definitions:

Definition. A **experiment** is the situation in which the results of the experiment are used to determine the probability.

Example. For Geoff rolling the die, the experiment is the rolling of the die.

Definition. An **outcome** of an experiment is the result of a single trial or iteration of the experiment.

Example. In rolling a die, there are six possible outcomes - 1,2,3,4,5, or 6.

Definition. The **sample space** of an experiment is the set of all possible outcomes.

Example. In rolling a die, the sample space is $\{1, 2, 3, 4, 5, 6\}$.

Definition. An **event** of an experiment is a collection of outcomes of the experiment.

Example. In rolling a die, an event could be that an even number is rolled. We can write this as a subset of the sample space - $\{2, 4, 6\}$.

Suppose we have an event A (such as A = rolling a 6 or A = rolling an even number). Then the probability of A, which we denote as $P(A)$ is

$$P(A) = \frac{\text{the number of ways } A \text{ could happen}}{\text{the size of the sample space}}$$

Example. Suppose our experiment is rolling a die once. Let A be the event of rolling a 6. There is only one way we can roll a 6. There are six elements in our sample space. Therefore

$$\begin{aligned} P(A) &= \frac{\text{the number of ways } A \text{ could happen}}{\text{the size of the sample space}} \\ &= \frac{1}{6} \end{aligned}$$

Example. Suppose our experiment is rolling a die once. Let B be the event of rolling an even number. There are three possible ways for us to roll an even number - either roll 2, 4, or 6. There are six elements in our sample space. Therefore

$$\begin{aligned} P(B) &= \frac{\text{the number of ways } B \text{ could happen}}{\text{the size of the sample space}} \\ &= \frac{3}{6} \\ &= \frac{1}{2} \end{aligned}$$

The Sample Space

Sometimes it can be a bit tricky to determine the sample space of an experiment. Here are some examples:

1) Suppose we are rolling a six-sided die. Then the sample space is $\{1, 2, \cdots, 6\}$.

2) Suppose the outcome of an experiment is the order in which five teams finish their math league relay. Call the teams $1, 2, 3, 4$, and 5. Then we can write possible outcomes as a permutation of the numbers 1 through 5. For example, $(2, 1, 5, 3, 4)$ would mean that team 2 is first, team 1 is second, team 5 is third, team 3 is fourth, and team 4 is fifth. Thus the sample space is all arrangements of the digits 1 through 5, of which there are 5! different possibilities.

3) Suppose we have an urn with 89 different balls inside, each ball uniquely labelled with a number 1 through 89, and our experiment is drawing at random a ball from the urn and recording its number. Then the sample space is $\{1, 2, \cdots, 89\}$ where each ball is represented by the number written upon it.

4) Suppose now we are rolling two six-sided dice. Then the sample space consists of 36 different items:

$$\{(i, j) \mid i \text{ is the value of the first die and } j \text{ is the value of the second die}\}$$

Complements

Suppose A is some event occurring. Let A^c denote that event not occurring.

Example. Suppose our experiment is flipping a coin three times is a row. Let A be the event that we get three heads in a row. Then A^c is the event that we do not get three heads in a row.

There is a nice formula relating the $P(A)$ and the $P(A^c)$, namely

$$1 = P(A) + P(A^c)$$

Sometimes it is easier to work with the complement of the event than with the event itself.

Some Worked Questions

These questions are from Sheldon Ross, *A First Course in Probability.* Answers are after the statement of the questions.

1) Suppose our experiment is flipping a coin three times in a row. Let B be the event that we do not get three heads in a row. Find $P(B)$.

2) An elementary school is offering three language classes: one is Spanish, one in French, and one in German. These classes are open to any of the 100 students in the school. There are 28 students in the Spanish class, 26 in the French class, and 16 in the German class. There are 12 students that are in both Spanish and French, 4 that are in both Spanish and German, and 6 that are in both French and German. In addition, there are 2 students taking all 3 classes.

   a) If a student is chosen randomly, what is the probability that he or she is not in any of these classes?

   b) If a student is chosen randomly, what is the probability that he or she is taking exactly one language class.

3) A deck of cards contains 52 cards. There are four suits and each suit contains 13 cards. A poker hand is a random dealing of five cards from the deck. What is the probability of being dealt

   a) a flush? (A hand is said to be a flush if all five cards are of the same suit)

   b) a pair of aces with no other repeated card values? (that is a hand a, a, b, c, d where a, b, c, and d are distinct and a is an ace).

   c) a pair with no other repeated card values? (that is a hand a, a, b, c, d where a, b, c and d are distinct).

   d) two pairs? (that is a hand with a, a, b, b, c where a, b, c and d are distinct).

   e) four of a kind? (that is a hand with a, a, a, a, b)

4) Two fair dice are rolled.

   a) What is the probability that the second die lands on a higher value than does the first?

   b) What is the probability that the sum of the values is a prime number?

   c) What is the probability the sum of the digits is a prime assuming the first dice rolled a value of either 3 or a 4.

5) An instructor gives her class a set of 10 problems with the information that the final exam will consist of a random selection of 5 of them. If a student has figured out how to do 7 of the problems and won't even attempt the other 3 if they appear on the exam, what is the probability that he will answer

a) all five problems?

b) at least four problems?

Solutions

1) Our sample space is all possible ways three coins could be tossed. There are two possibilities for the first toss (heads or tails), two possibilities for the second toss (heads or tails), and two possibilities for the third toss. Therefore there are 8 different possible outcomes (2^3), so the size of our sample space is 8.

Rather than finding the number of times we do not get three heads in a row, let's examine the complement, B^c when we get three heads in a row. There is only one outcome with three heads in a row - namely (heads, heads, heads).

Therefore,

$$\begin{aligned} P(B^c) &= \frac{\text{the number of ways } B^c \text{ could happen}}{\text{the size of the sample space}} \\ &= \frac{\text{the number of outcomes that is (heads, heads, heads)}}{8} \\ &= \frac{1}{8} \end{aligned}$$

Since

$$1 = P(B) + P(B^c)$$

we get

$$\begin{aligned} 1 &= P(B) + \frac{1}{8} \\ P(B) &= \frac{7}{8} \\ &= 0.875 \end{aligned}$$

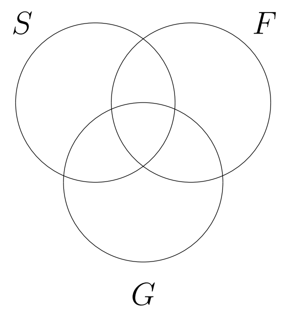
2) a) Our sample space is all the children in the school. There are 100 children, so the size of our sample space is 100.

Our event is that a student drawn at random is not taking any language classes. Call this event A

$$\begin{aligned} P(A) &= \frac{\text{the number of ways } A \text{ could happen}}{\text{the size of the sample space}} \\ &= \frac{\text{the number of students taking no language class}}{100} \end{aligned}$$

So we must find the number of students who are not taking any language class.

Let S be the number of students taking Spanish, F be the number of students taking French, and G be the number of students taking German. We draw a Venn diagram.



We know the following:

$$\begin{aligned} S &= 28 \\ F &= 26 \\ G &= 16 \\ S \cap F &= 12 \\ S \cap G &= 4 \\ F \cap G &= 6 \\ S \cap F \cap G &= 2 \end{aligned}$$

Then, using the formula

$$S + F + G - (S \cap F) - (S \cap G) - (F \cap G) + (S \cap F \cap G)$$

or working it out on the Venn diagram (put 2 in the centre, then work your way out), we get that there are 50 students who are taking language courses. Therefore there are 50 students who are not taking language courses.
Therefore

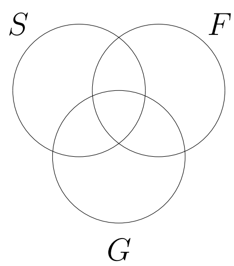
$$P(A) = \frac{50}{100} = \frac{1}{2} = 0.5$$

b) Our sample space is the same as in a).

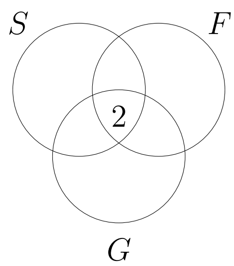
Let B be the event that a student chosen at random is taking exactly one language class. Then

$$\begin{aligned} P(B) &= \frac{\text{the number of ways } B \text{ could happen}}{\text{the size of the sample space}} \\ &= \frac{\text{the number of students taking exactly one language class}}{100} \end{aligned}$$

We need to work out the Venn Diagram from a).



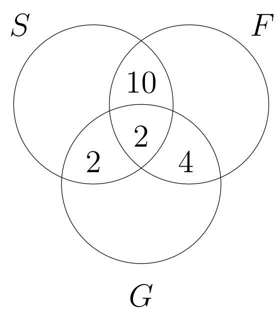
Since $S \cap F \cap G = 2$, we have



Then

$$\begin{aligned}(S\cap F)-(S\cap F\cap G) &= 12-2=10\\ (S\cap G)-(S\cap F\cap G) &= 4-2=2\\ (F\cap G)-(S\cap F\cap G) &= 6-2=4\end{aligned}$$

That is, the number of students taking Spanish and French, but not German is 10; the number of students taking Spanish and German but not French is 2; and the number of students taking French and German but not Spanish is 4. Then



Look at S. We know that $S = 28$, but of these 28 students, 10 are taking Spanish and French, but not German, 2 are taking Spanish and German, but not French, and 2 are taking all three. Thus there are

$$28-10-2-2=14$$

students who are only taking Spanish.

Look at F. We know that $F = 26$, but of these 26 students, 10 are taking Spanish and French, but not German, 4 are taking French and German, but not Spanish, and 2 are taking all three. Thus there are

$$26-10-4-2=10$$

students who are taking only French.

Look at G. We know that $G = 16$, but of these 16 students, 2 are taking German and Spanish, but not French, 4 are taking French and German, but not Spanish, and 2 are taking all three. Thus there are

$$16-2-4-2=8$$

students who are taking only German.

Thus the number of students taking only one language class is the number of students taking only Spanish plus the number of students taking only German plus the number of students taking only French, which is 14+10+8 =32.

Therefore

$$\begin{aligned} P(B) &= \frac{\text{the number of students taking exactly one language class}}{100} \\ &= \frac{32}{100} \\ &= \frac{8}{25} \\ &= 0.32 \end{aligned}$$

3) The sample space for all these events is the same. There are $\binom{52}{5} =$ 2 598 960 different ways to deal five cards from a deck of 52. Thus the size of our sample space is 2 598 960.

a) Let A be the event that a flush is dealt. Then

$$\begin{aligned} P(A) &= \frac{\text{the number of ways } A \text{ could happen}}{\text{the size of the sample space}} \\ &= \frac{\text{the number of hands which are a flush}}{2\ 598\ 960} \end{aligned}$$

Look at a suit - say $\heartsuit$'s. There are thirteen $\heartsuit$'s in a deck, and there are $\binom{13}{5}$ different ways to choose 5 cards from the thirteen. Thus there are $\binom{13}{5}$ different ways to be dealt 5 $\heartsuit$.

Similarly, there are $\binom{13}{5}$ different ways to be dealt 5 $\clubsuit$'s, 5 $\spadesuit$'s, or 5 $\diamondsuit$'s. Thus the number of ways to be dealt 5 cards from the same suit is

$$\binom{13}{5} + \binom{13}{5} + \binom{13}{5} + \binom{13}{5} = 5148$$

Then

$$P(A) = \frac{\text{the number of hands which are a flush}}{2\ 598\ 960} = \frac{5148}{2\ 598\ 960} \sim 0.002$$

b) Let B be the event that a pair of aces and no other duplicates is dealt. Then

$$\begin{aligned} P(B) &= \frac{\text{the number of ways } B \text{ could happen}}{\text{the size of the sample space}} \\ &= \frac{\text{the number of hands with a pair of aces and no other duplicates}}{2\,598\,960} \end{aligned}$$

We need to construct a hand with a pair of aces and no other duplicates.

There are 4 aces in a deck. We need to choose 2 of them; that is $\binom{4}{2}$.

There are now 12 other values in the deck: 2 through K. All our remaining cards must be chosen from these values. Moreover, no two cards can have the same value, otherwise they would be a pair. Thus, choose 3 different values, $\binom{12}{3}$, and from each of the values chosen, choose one card from that value $\binom{4}{1}$. That is, the number of hands with a pair of aces and no other duplicates is

$$\binom{4}{2}\binom{12}{3}\binom{4}{1}\binom{4}{1}\binom{4}{1} = 84480$$

Then

$$P(B) = \frac{84480}{2\,598\,960} \sim 0.033$$

c) Let C be the event that a pair and no other duplicates is dealt. Then

$$\begin{aligned} P(C) &= \frac{\text{the number of ways } C \text{ could happen}}{\text{the size of the sample space}} \\ &= \frac{\text{the number of hands with a pair and no other duplicates}}{2\,598\,960} \end{aligned}$$

From b), we know that the number of hands of having a pair of aces and no other duplicates is 84480. But we could have used the same argument and get the number of hands for a pair of twos and no other duplicates or a pair of sixes and no other duplicates, or any one of the thirteen denominations is a pair with no other duplicates. That is, there are 84480 hands with a pair of aces and no other duplicates, 84480 hands with a pair of twos and no other duplicates, 84480 with a pair of threes and no other duplicates, $\cdots$, 84480 hands with a pair of Kings and no other duplicates. Thus the number of hands with a pair and no other duplicates is

$$13 \cdot 84480 = 1\ 098\ 240$$

where the 13 is from each of the different values.

Therefore

$$P(C) = \frac{1\ 098\ 240}{2\ 598\ 960} \sim 0.423$$

d) Suppose first that we want a hand with a pair of aces, a pair of Kings, and some other card that isn't an ace or a King. There are $\binom{4}{2}$ ways to choose a pair of aces, $\binom{4}{2}$ ways to choose a pair of King, and $\binom{11}{1}\binom{4}{1} = 44$ ways to pick a card that isn't an ace or a king (the $\binom{11}{1}$ is choosing one of the other 11 values and the 4 choose 1 is choosing the one card of that value).

Thus there are 1584 hands with a pair of aces, a pair of Kings, and some other card which isn't a King or an ace.

But this holds in general. Thus for a fixed a, b and c, all distinct, there are 1584 hands with 2 a's, 2 b's and one c. Like in c), we know want to determine how many ways we can choose a and b distinct. There are thirteen possible values, and we want two distinct ones. That is $\binom{13}{2}$. Thus we add 1584 $\binom{13}{2}$ times to get 123 552. That is, there are 123 552 hands with two distinct pairs and one other distinct card.

Let D be the event that we are dealt two distinct pairs and one other distinct card. Then

$$P(D) = \frac{123\ 552}{2\ 598\ 960} \sim 0.048$$

e) Suppose first that we want a hand with four aces and some other card. There are 48 possible such hands which we get through $\binom{4}{4}\binom{48}{1}$; there is only one way to get four aces, and we need to take one of the remaining 48 cards.

However, this is the same number for any other four of a kind. There are 13 different values, so we get that the number of hands with four of a kind is

$$13 \cdot 48 = 624$$

Let E be the event that four of a kind is dealt. Then

$$P(E) = \frac{624}{2\ 598\ 960} \sim 0.0002$$

4) First, we will write out the sample space.

(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)
(2,1)	(2,2)	(2,3)	(2,4)	(2,5)	(2,6)
(3,1)	(3,2)	(3,3)	(3,4)	(3,5)	(3,6)
(4,1)	(4,2)	(4,3)	(4,4)	(4,5)	(4,6)
(5,1)	(5,2)	(5,3)	(5,4)	(5,5)	(5,6)
(6,1)	(6,2)	(6,3)	(6,4)	(6,5)	(6,6)

where for (i, j) in the table, i represents the value of the first die and j represents the value of the second die. There are 36 pairs in the sample space, so our sample space has size 36.

a) Let A be the event that the second die lands on a higher value than does the first. That is, this is all events where $i < j$ for (i, j) a value in the above table. There are 15 such pairs. Thus

$$P(A) = \frac{15}{36} = \frac{5}{12} \sim 0.417$$

b) Rewrite our sample space in terms of the sums:

2	3	4	5	6	7
3	4	5	6	7	8
4	5	6	7	8	9
5	6	7	8	9	10
6	7	8	9	10	11
7	8	9	10	11	12

Then the prime numbers are in bold:

2	**3**	4	**5**	6	**7**
3	4	**5**	6	**7**	8
4	**5**	6	**7**	8	9
5	6	**7**	8	9	10
6	**7**	8	9	10	**11**
7	8	9	10	**11**	12

Let B be the event that the sum of the values is a prime. From our chart, there are 15 such occurrences. Therefore

$$P(B) = \frac{15}{36} = \frac{5}{12} \sim 0.417$$

c) We now have an additional assumption that forces us to change our sample space. We are assuming that the first die rolled is either a three or a four. That is, we will only be looking at the third and fourth rows of the chart:

(3,1)	(3,2)	(3,3)	(3,4)	(3,5)	(3,6)
(4,1)	(4,2)	(4,3)	(4,4)	(4,5)	(4,6)

Thus our sample space has 12 elements.

Examine now the sums of these 12 elements, with the prime numbers in bold:

4	**5**	6	**7**	8	9
5	6	**7**	8	9	10

There are 4 prime numbers assuming that the first roll was either a 3 or a 4. Let D be the event that the sum of the digits is a prime given that the first roll was either a 3 or a four. Then

$$P(D) = \frac{4}{12} = \frac{1}{3} \sim 0.333$$

5) There are ten problems and the exam will be set with five of those problem. That is, the size of the sample space is $\binom{10}{5} = 252$.

a) Let A be the event that the student answers all five questions. That is, the exam must have had five questions chosen from the seven he knew how to do. There are $\binom{7}{5} = 21$ such exams. Then

$$\begin{aligned} P(A) &= \frac{\text{the number of ways } A \text{ could happen}}{\text{the size of the sample space}} \\ &= \frac{\text{the number of exams with questions he knows how to do}}{252} \\ &= \frac{21}{252} \\ &= \frac{1}{12} \\ &\sim 0.083 \end{aligned}$$

b) Let B be the event that he answers at least four questions. That is, B is the event that he answers either five questions or four questions but not a fifth.

We already know that there are 21 exams where he can answer all five questions. How many exams are there where he can answer four questions but not the fifth? We can choose four questions from the seven he knows and one question from the three he doesn't. That is, the number of exams in which he can answer four questions but not the fifth is $\binom{7}{4}\binom{3}{1} = 105$.

Then the size of B is $21+105 = 126$. Thus

$$\begin{aligned} P(B) &= \frac{\text{the number of ways } B \text{ could happen}}{\text{the size of the sample space}} \\ &= \frac{1260}{252} \\ &= \frac{1}{2} \\ &= 0.5 \end{aligned}$$

Resources

Charles M. Grinstead and J. Laurie Snell, *Introduction to Probability*. Chapters 1 and 3. `http://www.dartmouth.edu/~chance/teaching_aids/books_articles/probability_book/amsbook.mac.pdf`

Sheldon Ross, *A First Course in Probability*. Prentice Hall. Chapters 1 and 2.

Mrs. Glosser's Math Goodies; Probability. http://www.mathgoodies.com/lessons/vol6/intro_probability.html

The Math Forum @ Drexel; Introduction to Probability. http://mathforum.org/dr.math/faq/faq.prob.intro.html